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On The Binding Number of Corona of Complete graph with Paths and their complements

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Abstract

The binding number of several interesting types of graphs has been studied extensively in the literature. The determination of the binding number of a graph G , in general seams to be a hard problem. So, in this paper we study the binding number of some special graphs. Precisely his paper determines the binding number of the corona of the complete graphs with the paths and their complements.

Keywords: Binding number of graphs, Corona of graphs, Complete graphs, Paths.

1. Introduction

Woodall 1973 defined the binding number of a graph as follows. If $S \subseteq V(G)$, we write the open neighbourhood of the set S as $N(S) = \bigcup_{v \in S} N(v)$. The binding number of a graph G, denoted by $b(G)$, is given by

$$b(G) = \min_{S \in F} \frac{|N(S)|}{|S|}, \text{ where } F = \{S \subseteq V(G) : S \neq \Phi, N(S) \neq V(G)\}.$$

Woodall 1973 determined the binging number of the complete graphs, complete bipartite graphs, paths and cycles.(Kane et al.,1981) computed the binding number for a variety of graph products. (Kittrel et al., 1994) found the binding number of cubic $K-$hallian graphs.

The corona(Shakhatreh et al.,2006) of two graphs G_1 and G_2 denoted by $G_1 o G_2$ is defined to be the graph G obtained by taking one copy of G_1 (which has n_1 vertices) and n_1 copies of G_2, and joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. If G_2 has n_2 vertices , then it follows from the definition of the corona that $G_1 o G_2$ has $n_1(1+n_2)$ vertices and in general $G_1 o G_2$ is not isomorphic to $G_2 o G_1$.

$K_n, K_{m,n}, P_n, C_n, cp(n)$ and rK_2 stand for complete graph, complete bipartite graph, path, cycle , cocktail party graph and Ladder graph . (AL-Tobaili, 2009) determined the binding number of $K_n o K_{m,n}$. (AL-Tobaili, 2008) determined the binding number of $K_n o K_n$, $rK_2 o K_n$, $cp(m) o K_n$, $P_n o K_n$ and $C_n o K_n$. Also,(AL-Tobaili, 2010) computed the binding number of $K_n o rK_2$ and $K_n o cp(n)$.

In this paper we continue the study of the binding numbers of the corona of graphs. Precisely we determine the binding number of the complete graphs with paths and their complements.

Proposition 1 (Woodall 1973). If G is a graph on n vertices with minimum degree δ, then

$$b(G) \le \frac{n-1}{n-\delta}.$$

2. Binding number of the corona of K_n and P_m

Let $V(K_n) = \{v_1, v_2, \cdots, v_n\}, n \ge 5$ and $V(P_m) = \{b_1, b_2, \cdots, b_m\}, m \ge 5$. Let $w \in V(K_n o P_m)$. Then $\deg(w) = m + n - 1$, if $w = v_i, 1 \le i \le n$ and

$$\deg(w) = \begin{cases} 2, \ if \ w = b_i \ or \ b_m, \\ 3, \ if \ w = b_i, \ i \ne 1, m. \end{cases}$$

Lemma 2.1 1 Let $m \ge n \ge 5$. Then $\forall\ X \subseteq V(K_n o P_m)$ such that $N(X) \ne V(K_n o P_m)$, $X \cap K_n \ne \Phi$ and $X \cap (\bigcup_{i=1}^{n} P_{mi}) = \Phi$,

$$\min \frac{|N(X)|}{|X|} = \frac{mn + n - m}{n-1}.$$

Proof: We have two cases:

(i) $|X \cap K_n| = 1$. Then $\dfrac{|N(X)|}{|X|} = m + n - 1.$

(ii) $|X \cap K_n| = l,\ 2 \le l \le n-1.$ $\dfrac{|N(X)|}{|X|} = \dfrac{n + lm}{l}.$

Let $l = n-1$. Then $\frac{|N(X)|}{|X|} = \frac{n}{n-1} + m.$

Clearly $\min\left\{\frac{mn+n-m}{n-1}, m+n-1\right\} = \frac{mn+n-m}{n-1}.$

Lemma 2.2: 1 Let $m \geq n \geq 5$. Then $\forall\ X \subseteq V(K_n o P_m)$ such that $N(X) \neq V(K_n o P_m)$, $X \cap K_n = \Phi$ and $X \cap (\bigcup_{i=1}^{n} P_{mi}) \neq \Phi$,

$$\min \frac{|N(X)|}{|X|} = \begin{cases} \frac{2mn+2n-m-1}{2mn-m}, & \text{if } m \text{ is even}, \\ 1, & \text{if } m \text{ is odd}. \end{cases}$$

Proof: Since we are searching for the minimum of $\frac{|N(X)|}{|X|}$ and since $|N(X)|$ is increasing by increasing $|X|$, so, we try to make $|N(X)|$ and $|X|$ as nearly as possible. This can be satisfied by the following cases:

Case 1. m is an even integer and X consists of all the vertices $b_i : i$ is even from each of some l $(1 \leq l \leq n)$ copies of P_m. So,

$|X| = \frac{lm}{2}$ and $|N(X)| = \frac{lm}{2} + l.$ Therefore

$$\frac{|N(X)|}{|X|} = \frac{m+2}{m}.$$

Case 2. m is an odd integer and X consists of all the vertices $b_i : i$ is odd from each of some l $(1 \leq l \leq n)$ copies of P_m. Then we get $|X| = l(\frac{m+1}{2})$ and $|N(X)| = l(\frac{m-1}{2}) + l.$ Thus $\frac{|N(X)|}{|X|} = 1.$

Case 3. Let $n = l + k, l \geq 1, k \geq 1$.

We consider two sub - cases :

(i) m is an even integer and without loss of generality let X consists of the whole vertices of the l copies of P_m and the vertices $b_i : i$ is even from each of the k copies of P_m. So, $|X| = ml + \frac{km}{2}$, $|N(X)| = ml + \frac{km}{2} + k.$ To, maximize X ,let $l = n-1$ and $k = 1.$ Thus

$$\frac{|N(X)|}{|X|}=\frac{2nm+2n-m-1}{2nm-m}<\frac{m+2}{m}.$$

For, if $\frac{2nm+2n-m-1}{2nm-m}>\frac{m+2}{m}\Rightarrow 1+\frac{2n-1}{m(2n-1)}>1+\frac{2}{m}\Rightarrow 1>2,$ a contradiction.

(ii) m is an odd integer and without loss of generality let X consists of the whole vertices of the l copies of P_m and the vertices $b_i : i$ is odd from each of the k copies of P_m. So, $|X|=ml+k\left(\frac{m+1}{2}\right)$, $|N(X)|=ml+l+k\left(\frac{m-1}{2}\right)+k.$ To, maximize X, let $l=n-1$and $k=1.$ Thus

$$\frac{|N(X)|}{|X|}=\frac{2nm-m+2n-1}{2nm-m+1}=\frac{2mn-m+1+2n-2}{2mn-m+1}=1+\frac{2(n-1)}{2mn-m+1}>1.$$

This completes the proof of Lemma 2.2.

Lemma 2.3: 1 Let $m\geq n\geq 5.$ Then $\forall\ X\subseteq V(K_n o P_m)$ such that $N(X)\neq V(K_n o P_m),\ X\cap K_n\neq\Phi$ and $X\cap(\bigcup_{i=1}^{n}P_{mi})\neq\Phi,$

$$\min\frac{|N(X)|}{|X|}=\begin{cases}\frac{mn+n-m}{mn+n-m-1}, & if\ m\ is\ even,\\ 1, & if\ m\ is\ odd.\end{cases}$$

Proof: We consider two cases:

Case(I): $X\cap K_n=\{v_f\}, f$ is fixed, $1\leq f\leq n.$ We have the following sub-cases:

(a) For some fixed $j, 1\leq j\leq n,\ X\cap P_{mj}\neq\Phi$ and $X\cap\left(\bigcup_{i=1}^{n}P_{mi}\right)=\Phi, i\neq j.$

(a.1) $f=j.$ Then $|X|=1+m, |N(X)|=n+m.$ Thus

$$\frac{|N(X)|}{|X|}=\frac{n+m}{m+1}.$$

(a.2) $f\neq j.$ Three sub-sub cases to be considered:

(a.2.1) $P_{mj}\subseteq X.$ Then $|X|=1+m, |N(X)|=n-1+m+m=2m+n-1.$ Therefore

$$\frac{|N(X)|}{|X|}=\frac{n+2m-1}{m+1}.$$

(a.2.2) m is an even integer and X consists the even vertices from the copy P_{mj}.

Then $|X| = 1 + \frac{m}{2}$, $|N(X)| = n - 1 + m + \frac{m}{2}$. Thus

$$\frac{|N(X)|}{|X|} = \frac{2n + 3m - 2}{m + 2}.$$

(a.2.3) m is an odd integer and X consists the odd vertices from the copy P_{mj}.

Then $|X| = 1 + \frac{m+1}{2}$, $|N(X)| = n - 1 + m + \frac{m-1}{2}$. Therefore

$$\frac{|N(X)|}{|X|} = \frac{3m + 2n - 3}{m + 3}.$$

(b) For some fixed $j, 1 \le j \le n$, $X \cap P_{mj} = \Phi$ and $X \cap \left(\bigcup_{i=1}^{n} P_{mi} \right) \ne \Phi, i \ne j.$

(b.1) $f = j.$

(b.1.1) X consists the whole vertices of some l $(1 \le l \le n-1)$ copies of P_m of $\left(\bigcup_{i=1}^{n} P_{mi} \right)$, $i \ne j.$ Then $|X| = 1 + ml$, $|N(X)| = n - 1 + ml + m.$ To maximize X, let $l = n - 1.$ Hnce $\frac{|N(X)|}{|X|} = \frac{nm + n - 1}{nm - m + 1}.$

(b.1.2) m is an even integer and X consists the even vertices of some l $(2 \le l \le n-1)$ copies of P_m of $\left(\bigcup_{i=1}^{n} P_{mi} \right)$, $i \ne j.$ Then $|X| = 1 + \frac{lm}{2}$, $|N(X)| = n - 1 + \frac{lm}{2} + m.$ To maximize X, let

$l = n - 1.$ So, $\frac{|N(X)|}{|X|} = \frac{nm + 2n + m - 2}{nm - m + 2}.$

(b.1.3) m is an odd integer and X consists the vertices $b_i : i$ is odd from each of some l $(2 \le l \le n-1)$ copies of P_m of $\left(\bigcup_{i=1}^{n} P_{mi} \right)$, $i \ne j.$ Then

$|X| = 1 + l\left(\frac{m+1}{2} \right)$, $|N(X)| = n - 1 + l\left(\frac{m-1}{2} \right) + m.$ To maximize X, let

$l = n - 1.$ Thus $\frac{|N(X)|}{|X|} = \frac{nm + n + m - 1}{nm + n - m + 1}.$

(b.2) $f \ne j.$

(b.2.1) $X \cap P_{mf} \ne \Phi.$

(b.2.1.1) X consists the whole vertices of some $l\ (1 \le l \le n-2)$ copies of P_m of $\left(\bigcup_{i=1}^{n} P_{mi}\right)$, $i \ne j$. Then $|X| = 1 + m + ml$, $|N(X)| = n + m + ml$. To maximize X, let $l = n-2$. Therefore $\frac{|N(X)|}{|X|} = \frac{nm+n-m}{nm-m+1}$.

(b.2.1.2) m is an even integer and X consists the even vertices of some $l\ (2 \le l \le n-2)$ copies of P_m of $\left(\bigcup_{i=1}^{n} P_{mi}\right)$, $i \ne j$. Then $|X| = 1 + \frac{lm}{2} + m$, $|N(X)| = n + \frac{lm}{2} l + m$. To maximize X, let $l = n-2$. Thus $\frac{|N(X)|}{|X|} = \frac{nm+2n}{nm+2}$.

(b.2.1.3) m is an odd integer and X consists the odd vertices from each of some $l\ (2 \le l \le n-2)$ copies of P_m of $\left(\bigcup_{i=1}^{n} P_{mi}\right)$, $i \ne j$. Then $|X| = 1 + l\left(\frac{m+1}{2}\right) + m$, $|N(X)| = n + l\left(\frac{m-1}{2}\right) + m$. To maximize X, let $l = n-2$. Hence $\frac{|N(X)|}{|X|} = \frac{nm+n+2}{nm+n}$.

(b.2.2) $X \cap P_{mf} = \Phi$.

(b.2.2.1) X consists the whole vertices of some $l\ (1 \le l \le n-2)$ copies of P_m of $\left(\bigcup_{i=1}^{n} P_{mi}\right)$, $i \ne j$. Then $|X| = 1 + ml$, $|N(X)| = n - 1 + m + ml$. To maximize X, let $l = n-2$. So, $\frac{|N(X)|}{|X|} = \frac{nm+n-m-1}{nm-2m+1}$.

(b.2.2.2)) m is an even integer and X consists the even vertices of some $l\ (2 \le l \le n-2)$ copies of P_m of $\left(\bigcup_{i=1}^{n} P_{mi}\right)$, $i \ne j$. Then $|X| = 1 + \frac{lm}{2}$, $|N(X)| = n - 1 + \frac{lm}{2} + m$. To maximize X, let $l = n-2$. Then $\frac{|N(X)|}{|X|} = \frac{nm+2n-2}{nm-2m+2}$.

(b.2.2.3) m is an odd integer and X consists the odd vertices from each of some $l\ (2 \le l \le n-2)$ copies of P_m of $\left(\bigcup_{i=1}^{n} P_{mi}\right)$, $i \ne j$. Then

$|X| = 1 + l\left(\frac{m+1}{2}\right), |N(X)| = n - 1 + l\left(\frac{m-1}{2}\right) + m.$ To maximize $X,$ let $l = n - 2.$ Then $\frac{|N(X)|}{|X|} = \frac{nm + n}{nm + n - 2m}.$

(c) $X \cap P_{mi} \neq \Phi, \forall i = 1, 2, \cdots, m, i \neq f.$

The union $\left(\bigcup_{i=1}^{n} P_{mi}\right), i \neq f.$contains $n-1$ copies of $P_m.$ Let us divided these $n-1$ copies of P_m into two groups l and $k.$ That is let $n - 1 = l + k.$

(c.1) $X \cap P_{mf} \neq \Phi.$

(c.1.1) Let m be an even integer and let without loss of generality that X consists the whole vertices of the l copies of P_m and also consists the even vertices from each of the k copies of P_m. Then $|X| = 1 + m + ml + \frac{km}{2}, |N(X)| = n + m + ml + \frac{km}{2}.$ Let $l = n - 2$ and $k = 1.$Then

$$\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m}{nm - m + 2}.$$

(c.1.2) Let m be an odd integer and let without loss of generality that X consists the whole vertices of the l copies of P_m and also consists the vertices $b_i : i$ is odd from each of the k copies of P_m. Then

$|X| = 1 + m + ml + k\left(\frac{m+1}{2}\right), |N(X)| = n + m + ml + k\left(\frac{m-1}{2}\right).$ Let $l = n - 2$ and $k = 1.$Then $\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m - 1}{2nm - m + 3}.$

(c.2) $X \cap P_{mf} = \Phi.$

(c.2.1) X consists the whole vertices of the copies l and $k.$Then

$|X| = 1 + ml + mk, |N(X)| = n - 1 + m + ml + mk.$ So,

$$\frac{|N(X)|}{|X|} = \frac{n - 1 + m(l + k)}{1 + m(l + k)}.$$

Let $l + k = n - 1.$Then $\frac{|N(X)|}{|X|} = \frac{nm + n - 1}{nm - m + 1}.$

(c.2.2) m is an even integer and the l copies contained in X and also X consists the even vertices from each of the k copies of P_m. Then

$|X| = 1 + ml + \frac{km}{2}$, $|N(X)| = n - 1 + ml + \frac{km}{2}$. Let $l = n-2$ and $k = 1$. Therefore

$$\frac{|N(X)|}{|X|} = \frac{2nm + 2n - 3m - 2}{2nm - 3m + 2}.$$

(c.2.3) m is an odd integer and X consists the whole vertices of the l copies of P_m and also consists the vertices $b_i : i$ is odd from each of the k copies of P_m. Then $|X| = 1 + ml + k\left(\frac{m+1}{2}\right)$, $|N(X)| = n - 1 + ml + k\left(\frac{m-1}{2}\right)$. Let $l = n-2$ and $k = 1$. Thus

$$\frac{|N(X)|}{|X|} = \frac{2nm + 2n - 3m - 3}{2nm - 3m + 3}.$$

Case(II): $X \cap K_n = \{v_1, v_2, \cdots, v_l\}, 2 \le l \le n-1$. Then $K_n \subseteq N(X)$. Divided the copies of P_m into two groups l and k. That is $n = l + k$, $l \ge 2, k \ge 1$.

(a) X does not contain any vertices from any copy of the k copies of P_m. Then we take $|X| = l + ml$, $|N(X)| = n + ml$. Let $l = n-1$ and $k = 1$. Then

$$\frac{|N(X)|}{|X|} = \frac{nm + n - m}{nm + n - m - 1}.$$

(b) X contains vertices from each copy of some k copies of P_m.

(b.1) m is an odd integer and X consists the vertices $b_i : i$ is odd from each of some k copies of P_m. Then $|X| = l + ml + k\left(\frac{m+1}{2}\right)$, $|N(X)| = n + ml + k\left(\frac{m-1}{2}\right)$. Let $l = n-1$ and $k = 1$. Therefore

$$\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m - 1}{2nm + 2n - 1}.$$

(b.1.2) m is an even integer and X consists the even vertices from each of some k copies of P_m. Then $|X| = l + ml + \frac{km}{2}$, $|N(X)| = n + ml + \frac{km}{2}$. Let $l = n-1$ and $k = 1$. Thus

$$\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m}{2nm + 2n - m - 2}.$$

(c) X consists the whole vertices of each copy of the k copies of P_m except one copy , say P_{mj} such that $P_{mj} \not\subset X$.

(c.1) m is an even integer and X consists the even vertices from the copy P_{mj}

Then $|X| = l + ml + (k-1)m + \frac{m}{2}$, $|N(X)| = n + ml + (k-1)m + \frac{m}{2}$. Let $l = n-2$ and $k = 2$. Then

$$\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m}{2nm + 2n - 2m - 4}.$$

(c.1.2) m is an odd integer and X consists the vertices $b_i : i$ is odd from the copy P_{mj}. Then $|X| = l + ml + (k-1)m + \frac{m+1}{2}$, $|N(X)| = n + ml + (k-1)m + \frac{m-1}{2}$. Let $l = n-2$ and $k = 2$. Then

$$\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m - 1}{2nm + 2n - m - 3}.$$

We collect all the results in case of m is an even together in the following claim.

Claim 2.1:

$$\min\left\{\begin{array}{l} \frac{2n+3m-2}{m+2}, \frac{mn+2n+m-2}{mn-m+2}, \frac{mn+2n}{mn+2}, \frac{mn+2n-2}{mn-2m+2}, \frac{2mn+2n-m}{2mn-m+2,}, \\ \frac{2mn+2n-3m-2}{2mn-3m+2}, \frac{2mn+2n-m}{2mn+2n-m-2} \end{array}\right\}$$

$$= \frac{2mn+2n-3m-2}{2mn-3m+2}.$$

Proof: We show that every element of the above set is greater than $\frac{2mn+2n-3m-2}{2mn-3m+2}$.

Suppose not, that is every element of the above set is less than $\frac{2mn+2n-3m-2}{2mn-3m+2}$.

Let us take $\frac{mn+2n+m-2}{mn-m+2} < \frac{2mn+2n-3m-2}{2mn-3m+2}$, implies

$$4mn^2 + 4m^2n + 12m < 2mn^2 + 6m^2 + 8mn$$

$$\Rightarrow n^2 + 2mn + 6 < 3m + 4m, \text{ a contradiction , since}$$

$m \geq n \geq 5$.

In a similar manner we can show for the rest elements of the above set.

Similarly we collect all the results in case of m is an odd together in the following claim

Claim2.2:

$$\min\left\{\begin{array}{l}1,\dfrac{2mn+2n-m-1}{2mn-m+3},\dfrac{mn+n+2}{mn+n},\dfrac{mn+n}{mn+n-2m}.\dfrac{3m+2n-3}{m+3},\\ \dfrac{2mn+2n-3m-3}{2mn-3m+3},\dfrac{mn+n+m-1}{mn+n-m+1},\dfrac{2mn+2n-m-1}{2mn+2n-m-3}\end{array}\right\}$$
$$=1.$$

Proof: Proof is similar to the proof of Claim 2.1.

Also, collect all the results in case of there is no restriction on m together in the following claim.

Claim 2.3:

$$\min\left\{\frac{m+n}{m+1},\frac{2m+n-1}{m+1},\frac{mn+n-m}{mn-m\_1},\frac{mn+n-m-1}{mn-2m+1},\frac{mn+n-1}{mn-m+1},\frac{mn+n-m}{mn+n-m-1}\right\}$$
$$=\frac{mn+n-m}{mn+n-m-1}.$$

Proof: Proof is similar to the proof of Claim 2.1.

From Claim 2.1 and Claim 2.3, we have

Claim 2.4:

$$\min\left\{\frac{2mn+2n-3m-2}{2mn-3m+2},\frac{mn+n-m}{mn+n-m-1}\right\}=\frac{mn+n-m}{mn+n-m-1}.$$

Proof: Proof is similar to the proof of Claim 2.1.

Also, by Claim2.2 and Claim 2.3, we have

Claim 2.5: $\min\left\{1,\dfrac{mn+n-m}{mn+n-m-1}\right\}=1.$

This completes the proof of Lemma 2.3.

By Lemma 2.1, Lemma 2.2 and Claim 2.4 , we have

Claim 2.6:

$$\min\left\{\frac{mn+n-m}{n-1},\frac{2mn+2n-m-1}{2mn-m},\frac{mn+n-m}{mn+n-m-1}\right\}=\frac{mn+n-m}{mn+n-m-1}.$$

Proof: Proof is similar to the proof of Claim 2.1.

Also, by Lemma 2.1, Lemma 2.2 and Claim 2.5 , we have

Claim 2.7:

$$\min\left\{\frac{mn+n-m}{n-1}, \frac{2mn+2n-m-1}{2mn-m}, \frac{2mn+2n-m-1}{2mn+2n-1}\right\} = \frac{2mn+2n-m-1}{2mn+2n-1}.$$

Proof: Proof is similar to the proof of Claim 2.1.

Theorem 2.1 Let $m \geq n \geq 5.$ Then

$$b(K_n o P_m) = \begin{cases} \dfrac{mn+n-m}{mn+n-m-1}, & if\ m\ is\ even, \\ 1, & if\ m\ is\ odd. \end{cases}$$

Proof: By Lemma 2.1, Lemma 2.2, Lemma 2.3, Claim 2.6 and Claim 2.7, theorem follows.

3. Binding number of the corona of K_n and $\overline{P_m}$

Clearly two vertices $b_i, b_j \in V(\overline{P_m})$ $(1 \leq i, j \leq m)$ are nonadjacent if and only if $|i-j| = 1$ and adjacent if and only if $|i-j| \geq 2.$ Also,

$$|N(\{b_i, b_j\})| = \begin{cases} m-2, \ if & |i-j| = 1, \\ m-1, \ if & |i-j| = 2, \\ m, \ if & |i-j| \geq 3. \end{cases}$$

and $|N(\{b_i, b_{i+1}, b_{i+2}\})| = m - 1 (1 \leq i \leq m).$

Lemma 3.1 Let $m \geq n \geq 7.$ Then

$\forall\ X \subseteq V(K_n o \overline{P_m})$ such that $N(X) \neq V(K_n o \overline{P_m}),$ $X \cap K_n \neq \Phi$ and $X \cap \left(\bigcup_{i=1}^{n} \overline{P_{mi}}\right) = \Phi,$ $\min \dfrac{|N(X)|}{|X|} = \dfrac{mn+n-m}{n-1}.$

Proof: Proof is similar to the proof of Lemma 2.1.

Lemma 3.2 Let $m \geq n \geq 7.$ Then $\forall\ X \subseteq V(K_n o \overline{P_m})$ such that $N(X) \neq V(K_n o \overline{P_m}),$ $X \cap K_n = \Phi$ and $X \cap \left(\bigcup_{i=1}^{n} \overline{P_{mi}}\right) \neq \Phi,$ $\min \dfrac{|N(X)|}{|X|} = \dfrac{mn+n-1}{mn-n+3}.$

Proof: We consider the following cases:

(a) X consists only three vertices of the form b_i, b_{i+1}, b_{i+2} from each of some $l(1 \le l \le n)$ copies of $\overline{P_m}$. Then $|X| = 3l$, $|N(X)| = l(m-1)+l$. Thus

$$\frac{|N(X)|}{|X|} = \frac{m}{3}.$$

(b) X consists at least two vertices b_i, b_j such that $|i-j| \ge 3$ from each of some $l(1 \le l \le n)$ copies of $\overline{P_m}$. Then $|X| = ml$, $|N(X)| = ml + l$. Thus

$$\frac{|N(X)|}{|X|} = \frac{m+1}{m}.$$

(c) X consists at least two vertices b_i, b_j such that $|i-j| \ge 3$ from each some l copies of $\overline{P_m}$ and X consists only three vertices of the form b_i, b_{i+1}, b_{i+2} $(1 \le i \le m)$ from each of some $k(2 \le l+k \le n, l, k \ge 1)$ copies of $\overline{P_m}$. Then $|X| = ml + 3k$, $|N(X)| = ml + l + k(m-1) + k$. Let $l = n-1$ and $k = 1$. Thus

$$\frac{|N(X)|}{|X|} = \frac{mn+n-1}{mn-n+3}.$$

It can be easy seen that $\min\left\{\frac{m}{3}, \frac{m+1}{m}, \frac{mn+n-1}{mn-n+3}\right\} = \frac{mn+n-1}{mn-n+3}.$

Lemma 3.3 Let $m \ge n \ge 7$. Then $\forall\ X \subseteq V(K_n o \overline{P_m})$ such that $N(X) \ne V(K_n o \overline{P_m})$, $X \cap K_n \ne \Phi$ and $X \cap \left(\bigcup_{i=1}^{n} \overline{P_{mi}}\right) \ne \Phi$, $\min \frac{|N(X)|}{|X|} = \frac{mn+n-m}{mn+n-m-1}.$

Proof: Consider two cases:

Case(I): $X \cap K_n = \{v_f\}$, for some fixed $f, 1 \le f \le n.$

(I.1) $X \cap \overline{P_{mj}} \ne \Phi$, for some fixed $j, 1 \le j \le m$ and $X \cap \left(\bigcup_{i=1}^{n} \overline{P_{mi}}\right) = \Phi$, $i \ne j.$

(I.1.1) $f = j$. Then $|X| = m+1$, $|N(X)| = m+n$. Thus

$$\frac{|N(X)|}{|X|} = \frac{m+n}{m+1} > \frac{mn+n-m}{mn+n-m-1} \qquad \text{......(1).}$$

For, if

$$\frac{m+n}{m+1} < \frac{mn+n-m}{mn+n-m-1},$$
$$\Rightarrow mn^2 + n^2 - mn - n < mn + n$$
$$\Rightarrow mn - m + (n-1) < m+1$$
$$\Rightarrow mn < 2m+1$$
$$\Rightarrow n < 2 + \frac{1}{m}$$

$\Rightarrow n < 3,$ a contradiction.

(I.1.2) $f \neq j.$

(I.1.2.1) X consists only three consecutive vertices from the copy $\overline{P_{mf}}$.

Then $|X| = 1+3, |N(X)| = n-1+m-1+m.$ Thus

$$\frac{|N(X)|}{|X|} = \frac{2m+n-2}{4} > \frac{mn+n-m}{mn+n-m-1} \qquad \text{......(2).}$$

(I.1.2.2) X consists at least two vertices b_x, b_y such that $|x-y| \geq 3$ from the copy $\overline{P_{mf}}$. Then $|X| = m+1, |N(X)| = n-1+m+m.$ Thus

$$\frac{|N(X)|}{|X|} = \frac{2m+n-1}{m+1} > \frac{mn+n-m}{mn+n-m-1} \qquad \text{......(3).}$$

(I.2) $X \cap \left(\bigcup_{i=1}^{n} \overline{P_{mi}}\right) \neq \Phi.$

(I.2.1) $\overline{P_{mf}} \cap X \neq \Phi.$

(I.2.1.1) X consists only three consecutive vertices from each of some $l(1 \leq l \leq n-1)$ copies other than the copy $\overline{P_{mf}}$. Then $|X| = 1+m+3l, |N(X)| = n+m+l(m-1).$ Let $l = n-1.$ Thus

$$\frac{|N(X)|}{|X|} = \frac{mn+1}{3n+m-2} > \frac{mn+n-m}{mn+n-m-1} \qquad \text{......(4).}$$

(I.2.1.2) X consists at least two vertices b_x, b_y such that $|x-y| \geq 3$ from each of some $l(1 \leq l \leq n-2)$ copies other than the copy $\overline{P_{mf}}$. Then $|X| = 1+m+lm, |N(X)| = n+m+lm.$ Let $l = n-2.$ For if, $l = n-1,$ implies that

$N(X)=V(K_n o \overline{P_m})$. Thus

$$\frac{|N(X)|}{|X|}=\frac{mn+n-m}{mn-m+1}>\frac{mn+n-m}{mn+n-m-1} \qquad ......(5).$$

(I.2.1.3) X consists at leas two vertices b_x, b_y such that $|x-y|\geq 3$ from each of some l copies of $\overline{P_m}$ and X consists only three consecutive vertices from each of some k copies of $\overline{P_m}$ $(2\leq l+k\leq n-1,\ l,k\geq 1)$. Then
$|X|=1+lm+3k,\ |N(X)|=n+lm+k(m-1)$. Let $l=n-2$ and $k=1$. Thus

$$\frac{|N(X)|}{|X|}=\frac{mn+n-m-1}{mn-m+4}>\frac{mn+n-m}{mn+n-m-1} \qquad ......(6).$$

(I.3) $\overline{P_{mf}}\cap X=\Phi$.

(I.3.1) X consists at leas two vertices b_x, b_y such that $|x-y|\geq 3$ from each of some $l(1\leq l\leq n-1)$ copies of $\overline{P_m}$. $|X|=1+lm,\ |N(X)|=n-1+lm$. Let $l=n-1$. Thus

$$\frac{|N(X)|}{|X|}=\frac{mn+n-m-1}{mn-m+1}>\frac{mn+n-m}{mn+n-m-1} \qquad ......(7).$$

(I.3.2) X consists only three consecutive vertices from each of some l copies of $\overline{P_m}$ $(1\leq l\leq n-1)$. $|X|=1+3l,\ |N(X)|=n-1+l(m-1)$. Let $l=n-1$. Thus

$$\frac{|N(X)|}{|X|}=\frac{mn-m}{3n-2}>\frac{mn+n-m}{mn+n-m-1} \qquad ......(8).$$

(I.3.3) X consists at leas two vertices b_x, b_y such that $|x-y|\geq 3$ from each of some l copies of $\overline{P_m}$ and X consists only three consecutive vertices from each of some k copies of $\overline{P_m}$ $(2\leq l+k\leq n-1,\ l,k\geq 1)$. Then
$|X|=1+ml+3k,\ |N(X)|=n-1+ml+k(m-1)$. Let $l=n-2$ and $k=1$. Thus

$$\frac{|N(X)|}{|X|}=\frac{mn+n-m-2}{mn-2m+4}>\frac{mn+n-m}{mn+n-m-1} \qquad ......(9).$$

Case(II): $X\cap K_n=\{v_1,v_2,\cdots,v_l\},\ 2\leq l\leq n-1$. Then $K_n\subseteq N(X)$. Let there exist at least one vertex, say, $v_f\in V(K_n)$ and $v_f\notin K_n\cap X$, otherwise $N(X)=V(K_n o \overline{P_m})$. Clearly the vertices $v_1,v_2,\cdots,v_l$ are joined to the copies $\overline{P_{m1}},\overline{P_{m2}},\cdots,\overline{P_{ml}}$ respectively. Let there are k copies of $\overline{P_m}$ that are different from the $\overline{P_{m1}},\overline{P_{m2}},\cdots,\overline{P_{ml}}$

$(2 \le l+k \le n-1)$ copies. Consider the copy $\overline{P_{mf}}$.

(II.1) X consists at least two vertices b_x, b_y such that $|x-y| \ge 3$ from each of the k copies of $\overline{P_m}$.

(II.1.1) $X \cap \overline{P_{mf}} = \Phi$. Then $|X| = l + ml + mk$, $|N(X)| = n + ml + mk$.

(II.1.1.1) $k = 0$. Thus

$$\frac{|N(X)|}{|X|} = \frac{n+ml}{l+ml}.$$ To maximize X, let $l = n-1$. So,

$$\frac{|N(X)|}{|X|} = \frac{mn+n-m}{mn+n-m-1} \qquad ......(10).$$

(II.1.1.2) $k \ge 1$. Then $k = n-l-1$. Therefore

$$\frac{|N(X)|}{|X|} = \frac{n+mn-m}{mn-m+l}.$$ To maximize X, let $l = n-2$. Hence

$$\frac{|N(X)|}{|X|} = \frac{n+mn-m}{mn-m+n-2} > \frac{mn+n-m}{mn+n-m-1} \qquad ......(11).$$

(II.1.2) $X \cap \overline{P_{mf}} = \{b_i, b_{i+1}, b_{i+2}\}$. Then
$|X| = l + ml + mk + 3$, $|N(X)| = n + ml + mk + m - 1$.

(II.1.2.1) $k = 0$. Take $l = n-1$. Then

$$\frac{|N(X)|}{|X|} = \frac{mn+n-1}{mn+n-m+2} > \frac{mn+n-m}{mn+n-m-1} \qquad ......(12).$$

(II.1.2.2) $k \ge 1$. Then as in (II.1.1.2), we get

$$\frac{|N(X)|}{|X|} = \frac{mn+n-1}{mn+n-m+1} > \frac{mn+n-m}{mn+n-m-1} \qquad ......(13).$$

(II.2) X consists only three consecutive vertices from each of the k copies of $\overline{P_m}$.

(II.2.1) $X \cap \overline{P_{mf}} = \Phi$. Then $|X| = l + ml + 3k$, $|N(X)| = n + ml + k(m-1)$. Let $l = n-2$ and $k = 1$. Thus

$$\frac{|N(X)|}{|X|} = \frac{mn+n-m-1}{mn-n-2m+1} > \frac{mn+n-m}{mn+n-m-1} \qquad ......(14).$$

$X \cap \overline{P_{mf}} = \{b_i, b_{i+1}, b_{i+2}\}$. Then

$|X| = l + ml + 3k + 3,\ |N(X)| = n + ml + k(m-1) + m - 1.$ Let $l = n-2$ and $k = 1$.

Thus $\dfrac{|N(X)|}{|X|} = \dfrac{mn+n-2}{mn+n-2m+4} > \dfrac{mn+n-m}{mn+n-m-1}$ ......(15).

Case(II.3) Let $k = a+b, a \geq 1, b \geq 1.$ Let X consists at least two vertices b_x, b_y such that $|x-y| \geq 3$ from each of the a copies and only three consecutive vertices form each of the b copies of $\overline{P_m}$. Thus

$|X| = l + ml + ma + 3b,\ |N(X)| = n + ml + ma + b(m-1).$ To maximize $X,$ let $a = b = 1.$ Then

$\dfrac{|N(X)|}{|X|} = \dfrac{mn+n-m-1}{mn+n-2m} > \dfrac{mn+n-m}{mn+n-m-1}$ ......(16).

The proofs of (2) up to (16) is similar to (1).

Thus by (1) up to (16), Lemma 3.3 follows.

From Lemma 3.1, Lemma 3.2 and Lemma 3.3, we have the following claim.

Claim 3.1 $\min\left\{\dfrac{mn+n-m}{n-1}, \dfrac{mn+n-1}{mn-n+3}, \dfrac{mn+n-m}{mn+n-m-1}\right\} = \dfrac{mn+n-m}{mn+n-m-1}.$

Proof: Assume on the contrary that $\dfrac{mn+n-m}{n-1} < \dfrac{mn+n-m}{mn+n-m-1}$

$$\Rightarrow m^2n^2 - 2m^2n + mn^2 - mn + m^2 < 0$$
$$\Rightarrow m^2n^2 - 2m^2n + mn^2 - mn < 0$$
$$\Rightarrow m^2n^2 + mn^2 < 2m^2n + mn$$
$$\Rightarrow mn + n < 2m + 1$$
$$\Rightarrow mn < 2m + 1$$
$$\Rightarrow n < 2 + \frac{1}{m}$$
$$\Rightarrow n < 3,$$

A contradiction.

Similarly we can show that $\dfrac{mn+n-1}{mn-n+3} > \dfrac{mn+n-m}{mn+n-m-1}.$

Hence the claim is proved.

Theorem 3.1 Let $m \geq n \geq 7.$ Then

$$b(K_n o \overline{P_m}) = \frac{mn+n-1}{mn+n-m-1}.$$

Proof: For small values of m and n the theorem can easy to prove**.** Let $m \geq n \geq 7$. Then by Lemma 3.1, Lemma 3.2, Lemma 3.3, and Claim 3.1, theorem follows.

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