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Three Step Iteration Process with Errors in Convex Cone Metric Spaces

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Abstract

The object of this paper is to consider a Noor type iteration process with errors, which approximate the fixed point of two asymptotically quasi-nonexpansive mappings in convex cone metric spaces. Our results also extend, improve and generalize many known results from the existing literature.

AMS subject classification: 47H10, 54H25.
Keywords: Three step iteration process with errors, cone metric space, asymptotically quasi-nonexpansive mapping, common fixed point, normal and no-normal cone.

1. Introduction

In 1967, Diaz and Metcalf [4] introduced the concept of quasi-nonexpansive mappings. In 1972, Goebel and Kirk [5] introduced the concept of asymptotically nonexpansive mappings. The convergence of Ishikawa iterates of asymptotically quasi-nonexpansive mappings on convex metric spaces is obtained in [22]. Convergence theorems for some iterates of nonexpansive mappings, quasi-nonexpansive mappings and their generalized types have been proved in metric and Banach spaces (see, e.g., [2, 1, 7, 8, 23, 29, 30, 33]). The convergence of certain iterative schemes of a finite family of asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings in Banach spaces is established in [3, 7]. Three-step methods (named as Noor methods by some authors) for solving various classes of variational inequalities and related problems were extensively studied by the same author in [11].

In 1989, Glowinski and Le-Tallec [18] used a three-step iterative process to solve elastoviscoplasticity, liquid crystal and eigenvalue problems. They established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge et al. [19] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [18] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iteration also lead to highly parallelized algorithms under certain conditions. Hence, we can conclude by observing that three-step iterative scheme play an important role in solving various problems in pure and applied sciences. Studies in nonlinear functional analysis reveals that several problems in sciences, engineering and management sciences can be converted and solved as a fixed point problem of the form x = Tx; where T is a mapping. Several authors in literature have obtained some interesting fixed points results (see, e.g. [9, 22, 33, 10, 11, 16, 12, 15]).

In 2002, Xu and Noor [33] considered a three-step iterative scheme with fixed-point iterations for asymptotically nonexpansive mappings in Banach spaces. In 2007, Noor and Huang [13] analyzed three-step iteration methods for finding the common element of the set of fixed points of nonexpansive mappings and studied the convergence criteria for three-step iterative methods. In 2007, Nammanee and Suanti [14] considered the weak and strong convergences for asymptotically nonexpansive mappings for the modified Noor iteration schemes with errors in a uniformly convex Banach spaces. In 2008, Khan et al. [7] generalized the Noor-type iterative process considered in [33] to the case of a finite family of mappings. In 2010, Khan and Ahmed [24] introduced a generalized iterative scheme due to Khan et al. [7] in convex metric spaces and established a strong convergence to a unique common fixed point of a finite family of asymptotically quasi-nonexpansive mappings under the scheme. Very recently, Tian and Yang [27] gave some sufficient and necessary conditions for a new Noor-type iteration with errors to approximate a common fixed point for a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces.

The Picard and Mann [21] iteration schemes for a mapping $T : K \to K$ are defined by

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = T x_n \end{cases} \tag{1.1}$$

and

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, n \in \mathbb{N} \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is in $[0, 1]$. It is well-known that Picard iteration scheme converges for contractions but not converges for nonexpansive mapping whereas Mann iteration scheme converges for nonexpansive.

The Ishikawa's iteration process [17] which is defined recursively by

$$\begin{cases} x_0 = x \in K, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, n \in \mathbb{N} \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0,1]$.

The Noor iteration process is defined by the sequence $\{x_n\}$.

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = \alpha_n Ty_n + (1-\alpha_n)x_n; \\ y_n = \beta_n Tz_n + (1-\beta_n)x_n; \\ z_n = \gamma_n Tx_n + (1-\gamma_n)x_n, n \in \mathbb{N} \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0,1]$.

In 2011, Phuengrattana and Suantai [20] introduced the following modified three-step SP-iteration process and used it to approximating a fixed point of continuous functions on an arbitrary interval.

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = \alpha_n Ty_n + (1-\alpha_n)y_n; \\ y_n = \beta_n Tz_n + (1-\beta_n)z_n; \\ z_n = \gamma_n Tx_n + (1-\gamma_n)x_n, n \in \mathbb{N} \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[0,1]$.

Now we define the above iterative schemes in the setting of convex cone metric spaces:

Let (X, d, W) be a convex cone metric spaces. For any $x_0 \in X$, we have

1.1. Picard and Mann iterative scheme

$$x_{n+1} = Tx_n, n \in \mathbb{N}, \tag{1.6}$$

and

$$x_{n+1} = W(x_n, Tx_n, \alpha_n), \tag{1.7}$$

where $n \in \mathbb{N}$ and $\{\alpha_n\}$ is sequence in the interval $[0,]$.

1.2. Ishikawa iterative scheme

$$\begin{aligned} x_{n+1} &= W(x_n, Ty_n, \alpha_n) \\ y_n &= W(x_n, Tx_n, \beta_n), \end{aligned} \tag{1.8}$$

where $n \in \mathbb{N}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0,1]$.

1.3. Noor iterative scheme

$$
\begin{aligned}
x_{n+1} &= W(x_n, Ty_n, \alpha_n) \\
y_n &= W(x_n, Tz_n, \beta_n), \\
z_n &= W(x_n, Tx_n, \gamma_n),
\end{aligned}
\tag{1.9}
$$

where $n \in \mathbb{N}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0, 1]$.

1.4. Modified Three-step SP-iteration Process

$$
\begin{aligned}
x_{n+1} &= W(y_n, Ty_n, \alpha_n), \\
y_n &= W(z_n, Tz_n, \beta_n), \\
z_n &= W(x_n, Tx_n, \gamma_n),
\end{aligned}
\tag{1.10}
$$

where $n \in \mathbb{N}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[0, 1]$.

The purpose of this paper is to prove strong convergence theorems to approximate fixed point of the modified SP-iteration process for generalized asymptotically quasi-nonexpansive mapping in the framework of Convex cone metric spaces.

2. Preliminaries

Throughout this paper, E is a normed vector space with a normal solid cone P. A nonempty subset P of E is called a cone if P is closed, $P \neq \{0\}$, for $a, b \in R^+ = [0, \infty)$ and $x, y \in P$, $ax + by \in P$ and $P \cap (-P) = \{0\}$. We define a partial ordering $\preceq$ in E as $x \preceq y$ if $y - x \in P$. $x \preceq y$ indicates that $y - x \in int P$ and $x \prec y$ means that $x \preceq y$ but $x \neq y$. A cone P is said to be solid if $int P$ is nonempty.

There exist two kinds of cones-normal (with the normal constant k) and non-normal cones [6]). Let E be a real Banach space, $P \subset E$ is a cone and $\leq$ is partial ordering defined by P. A cone P is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies

$$
\|x\| \leq k\|y\|. \tag{2.1}
$$

for all $x, y \in P$.
or equivalently, if (for all n) $x_n \leq y_n \leq z_n$ and

$$
\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{2.2}
$$

The least positive number k satisfying (2.1) is called the normal constant of P.

Definition 2.1. Let X be a nonempty set. Suppose the mapping $d : X \times X \to {}'E$ satisfies:

(d1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space. It is clear that the concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = R$ and $P = [0, +\infty)$.

Definition 2.2. Let $\{x_n\}$ be a sequence in a cone metric space (X, d) and P be a normal cone with a normal constant K (see [6]). Then $\{x_n\}$ is called

(i) a Cauchy sequence if for every ϵ in E with $0 \ll \epsilon$, there is a natural number N such that for all $n, m > N$, $d(x_n, x_m) \ll \epsilon$;

(ii) a convergent sequence if for every ϵin E with $0 \ll \epsilon$, then there is a natural number N such that for all $n \geq N$, $d(x_n, x) \ll \epsilon$ for some fixed x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

We recall [6] that if P is a normal solid cone, then $\{x_n\} \in X$ is a Cauchy sequence if and only if $\|d(x_n, x_m)\| \to 0$, as $n, m \to \infty$. Moreover, $\{x_n\} \in X$ converges to $x \in X$ if and only if $\|d(x_n, x)\| \to 0$ as $n \to \infty$.

3. Convexity of Several Contractive Conditions in Cone Metric Space

Definition 3.1. Let X be a cone metric space and $T : X \to X$ be a mapping (see [5, 8]. Then

(i) T is called Nonexpansive if

$$d(Tx, Ty) \preceq d(x, y),$$

for all $x, y \in X$.

(ii) T is called Quasi-nonexpansive if $F(T) \neq \phi$ and

$$d(Tx, p) \preceq d(x, p),$$

for all $x, y \in X$, and $p \in F(T)$.

(iii) T is called asymptotically nonexpansive mapping if there exist a sequence $k_n \subset [0, \infty)$ with $\lim\limits_{n\to\infty} k_n = 0$ such that

$$(T^n x - T^n y) \preceq (1 + k_n)(x - y)$$

for all $x, y \in X$ and $n \geq 1$.

(iv) T is called quasi-nonexpansive mapping provided

$$(T^n x - p) \preceq (1 + k_n)(x - p)$$

for all $x \in X$, $p \in F(T)$ and $n \geq 1$.

From the above definition, if $F(T) \neq \phi$, it follows that an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive.

In recent years, asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings have been studied extensively in the setting of convex metric spaces ([26, 22, 25, 23]).

In 1970, Takahashi [31] first introduced a notion of convex metric space which is more general space. It should be pointed out that each linear normed space is a special example of convex metric space, but there exist some convex metric spaces which cannot be embedded into normed space [31].

Now, we give the following definitions which will be used in our main results:

Definition 3.2. Let $(X, d, W))$ be a cone metric space, and $I = [0, 1]$. A mapping $W : X^2 \times I \to X$ is said to be convex structure on X, if for any $(x, y, \lambda) \in X^2 \times I$ and $u \in X$, the following inequality holds:

$$d(W(x, y, \lambda), u) \preceq \lambda d(x, u) + (1 - \lambda) d(y, u).$$

If (X, d) is a cone metric space with a convex structure W, then (X, d) is called a convex abstract metric space or convex cone metric space (see also [14], [23]). Moreover, a nonempty subset K of X is said to be convex if $W(x, y, \lambda) \in K$, for all $(x, y, \lambda) \in K^2 \times I$.

Definition 3.3. Let $(X, d, W))$ be a cone metric space, $I = [0, 1]$, and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are real sequences in $[0, 1]$ with $a_n + b_n + c_n = 1$. A mapping $W : X^3 \times I^3 \to X$ is said to be convex structure on X, if for any $(x, y, z, a_n, b_n, c_n) \in X^3 \times I^3$ and $u \in X$, the following inequality holds:

$$d(W(x, y, z, a_n, b_n, c_n), u) \preceq a_n d(x, u) + b_n d(y, u) + c_n d(z, u).$$

If (X, d) is a cone metric space with a convex structure W, then (X, d) is called a generalized convex cone metric space. Furthermore, a nonempty subset K of X is said to be convex if $W(x, y, z, a_n, b_n, c_n) \in K$, for all $(x, y, z, a_n, b_n, c_n) \in K^3 \times I^3$.

Remark 3.4. If $E = R$, $P = [0, +\infty)$, $\|\cdot\| = |\cdot|$ then (X, d) is a convex metric space, i.e., generalized convex metric space as in [32].

Example 3.5.

(a) [6] Let $E = R^2$, $P = \{(x, y) \in R^2 : x \geq 0, y \geq 0$, $X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space [11] with normal cone P where $k = 1$.

(b) For other examples of a cone metric spaces, i.e., P-metric spaces one can see [34].

Example 3.6. Let (X, d) be a cone metric space as in Example **??**(a). If $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$, then (X, d) is a cone metric space. Therefore, this notion is more general than that of a convex metric space.

Definition 3.7. Let (X, d, W) be a cone metric space with a convex structure $W : X^3 \times I^3 \to X$ and $T : X \to X$ be a generalized asymptotically quasi-nonexpansive mappings with nine sequences of real numbers $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}, \{c_n^{'}\}, \{a_n^{''}\}, \{b_n^{''}\}$ and $\{c_n^{''}\}$in $[0, 1]$ with $a_n + b_n + c_n = 1$, $a_n^{'} + b_n^{'} + c_n^{'} = 1$, and $a_n^{''} + b_n^{''} + c_n^{''} = 1$ for $n \in N$.

In this paper, we consider the following modified SP-iteration process with errors in convex cone metric spaces.

For any given $x_1 \in X$, define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_0 = x \in K, \\ x_{n+1} = W(y_n, T^n y_n, u_n, a_n, b_n, c_n), \\ y_n = W(z_n, T^n z_n, v_n, a_n^{'}, b_n^{'}, c_n^{'}), \\ z_n = W(x_n, T^n x_n, w_n, a_n^{''}, b_n^{''}, c_n^{''}), \end{cases} \tag{3.1}$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are three sequences in X. Then a sequence $\{x_n\}$ is called a modified SP-iteration with errors for a generalized asymptotially quasi-nonexpansive mapping T in convex cone metric space (X, d).

In the consequence, we shall need the following lemma.

Lemma 3.8. (see [28].) Let $\{a_n\}$, $\{b_n\}$ and $\{\alpha_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} = (1 + a_n)a_n + b_n, n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$. Then
(a) $\lim_{n\to\infty} a_n$ exists.
(b) If $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

4. Main results

In this section, we propose more generalized convergence theorem regarding three step SP-iteration process with errors for approximating a common fixed point of a sequences of generalized asymptotically quasi-nonexpansive mappings in convex cone metric spaces.

Theorem 4.1. Let K be a nonempty, closed, convex subset of a complete convex cone metric space $X, T : K \to K$ be a generalized asymptotically quasi-nonexpansive mapping with a sequence $\{k_n\}$ and $\{r_n\} \in [0, \infty)$ such that $\sum_{n=1}^{\infty}\{k_n\} < \infty$ and $\sum_{n=1}^{\infty}\{r_n\} < \infty$. Assume that $F = F(T) \neq \phi$. Let $\{x_n\}$ be a SP-iteration process with errors defined by (3.1) and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in K. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$ and $\{c_n^{'}\}$ be six sequences in [0, 1] with restriction $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1$ and $\sum_{n=1\to\infty} (c_n + c_n^{'} + c_n^{''}) < \infty$ and if $F = F(T) \neq \phi$ then:

(i) there exists a constant vector $v \in P \setminus \{0\}$ such that

$$\|d(x_{n+1}, p)\| \preceq k \cdot (1 + \alpha_n) \cdot \|d(x_n, p)\| + k \cdot \beta_n + k \cdot \|v\| \cdot \gamma_n,$$

for all $n \in N$ and for all $p \in F$, where k is the normal constant of a cone P, where $v = \sup_{p\in F, n\geq 1} \{d(u_n, p) + (1 + k_n)d(v_n, p) + (1 + k_n)^2 d(w_n, p)\}$.

(ii) there exists a real constant $M > 0$ such that

$$\|d(x_{n+m}, p)\| \preceq k \cdot M \cdot d(x_n, p) + k \cdot M \cdot \sum_{i=1}^{n+m-1} \beta_i + \cdot N \cdot \Big(\sum_{i=1}^{n+m-1} \gamma_i\Big) \cdot v,$$

for all $n, m \in N$ and for all $p \in F$, where k is the normal constant of a cone P.

Proof. (i) We suppose that $p \in F$. Then, we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(W(y_n, T^n y_n, u_n, a_n, b_n, c_n), p) \\
&\preceq a_n d(y_, p) + b_n d(T^n y_n, p) + c_n d(u_n, p) \\
&\preceq a_n d(y_n, p) + b_n[(1 + k_n)d(y_n, p) + r_n] + c_n d(u_n, p) \\
&\preceq [a_n + b_n(1 + k_n)]d(y_n, p) + b_n r_n + c_n d(u_n, p) \\
&\preceq (a_n + b_n)(1 + k_n)d(y_n, p) + b_n r_n + c_n d(u_n, p) \\
&= (1 - c_n)(1 + k_n)d(y_n, p) + b_n r_n] + c_n d(u_n, p) \\
&\preceq (1 + k_n)d(y_n, p) + r_n + c_n d(u_n, p),
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
d(y_n, p) &= d(W(z_n, T^n z_n, v_n, a_n^{'}, b_n^{'}, c_n^{'}), p) \\
&\preceq a_n^{'} d(z_n, p) + b_n^{'} d(T^n z_n, p) + c_n^{'} d(v_n, p) \\
&\preceq a_n^{'} d(z_n, p) + b_n^{'}[(1 + k_n)d(z_n, p) + r_n] + c_n^{'} d(v_n, p) \\
&\preceq [a_n^{'} + b_n^{'}(1 + k_n)]d(z_n, p) + b_n^{'} r_n + c_n^{'} d(v_n, p) \\
&= (1 - c_n^{'})(1 + k_n)d(z_n, p) + b_n^{'} r_n + c_n^{'} d(v_n, p) \\
&\preceq (1 + k_n)d(z_n, p) + r_n + c_n^{'} d(v_n, p),
\end{aligned} \tag{4.2}$$

Moreover,

$$\begin{aligned}
d(z_n, p) &= d(W(x_n, T^n x_n, w_n, a_n^{''}, b_n^{''}, c_n^{''}), p) \\
&\preceq a_n^{''} d(z_n, p) + b_n^{''}[(1 + k_n)d(x_n, p) + r_n] + c_n^{''} d(w_n, p) \\
&\preceq [a_n^{''} + b_n^{''}(1 + k_n)d(x_n, p)] + b_n^{''} r_n c_n^{'} d(w_n, p) \\
&\preceq [a_n^{''} + b_n^{''}](1 + k_n)d(x_n, p) + b_n^{''} r_n c_n^{''} d(w_n, p) \\
&= (1 - c_n^{''})(1 + k_n)d(x_n, p) + b_n^{''} r_n c_n^{''} d(w_n, p) \\
&\preceq (1 + k_n)d(x_n, p) + r_n + c_n^{''} d(w_n, p).
\end{aligned} \tag{4.3}$$

Substituting (4.2) into (4.1), it can be obtained that

$$\begin{aligned}
d(x_{n+1}, p) &\preceq (1 + k_n)[(1 + k_n)d(z_n, p) + r_n + c_n^{'} d(v_n, p)] + r_n + c_n d(u_n, p) \\
&\preceq (1 + k_n)^2 d(z_n, p) + (1 + k_n)r_n + (1 + k_n)c_n^{'} d(v_n, p) \\
&\quad + r_n + c_n d(u_n, p).
\end{aligned} \tag{4.4}$$

Again, putting (4.3) into (4.4), we have

$$\begin{aligned}
d(x_{n+1}, p) &\preceq (1 + k_n)^2[(1 + k_n)d(x_n, p) + r_n + c_n^{''} d(w_n, p)] \\
&\quad + (1 + k_n)r_n + (1 + k_n)c_n^{'} d(v_n, p) + r_n + c_n d(u_n, p) \\
&\preceq (1 + k_n)^3 d(x_n, p) + (1 + k_n)^2 r_n + (1 + k_n)^2 c_n^{''} d(w_n, p)] \\
&\quad + (1 + k_n)r_n + (1 + k_n)c_n^{'} d(v_n, p) + r_n + c_n d(u_n, p) \\
&\preceq (1 + 3k_n + 3k_n^2 + k_n^3)d(x_n, p) + (2 + 3k_n + k_n^2)r_n \\
&\quad + (1 + k_n)^2 c_n^{''} d(w_n, p)] + (1 + k_n)c_n^{'} d(v_n, p) + c_n d(u_n, p).
\end{aligned}$$

If $\alpha_n = 3k_n + 3k_n^2 + k_n^3$,$\beta_n = (2 + 3k_n + k_n^2)r_n$, and $\gamma_n = c_n + c_n^{'} + c_n^{''}$. Since the sequence $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded, there exists an element $v \in P$ $\{0\}$ such that $\{(1 + k_n)^2 d(w_n, p) + (1 + k_n)d(v_n, p) + d(u_n, p)\} \preceq v$ for any $n \in N$ and $p \in P$. In this case, we can obtained

$$d(x_{n+1}, p) \preceq (1 + \alpha_n)d(x_n, p) + \beta_n + v\gamma_n. \tag{4.5}$$

This completes the proof of part (i).

(ii) Notice that $1 + x \preceq e^x$ for all $x \geq 0$. Using this and $\sum_{n=1}^{\infty} A_n < \infty$, we have

$$\begin{aligned} d(x_{n+m}, p) &\preceq (1 + \alpha_{n+m-1}) d(x_{n+m-1}, p) + \beta_{n+m-1} + v \cdot \gamma_{n+m-1} \\ &\preceq e^{\alpha_{n+m-1}} d(x_{n+m-1}, p) + \beta_{n+m-1} + v \cdot \gamma_{n+m-1} \\ &\preceq e^{\alpha_{n+m-1}} [e^{\alpha_{n+m-2}} d(x_{n+m-2}, p) + \beta_{n+m-2} + v \cdot \gamma_{n+m-2}] \\ &\quad + \beta_{n+m-1} + v \cdot \gamma_{n+m-1} \\ &\preceq e^{\alpha_{n+m-1}+\alpha_{n+m-2}} d(x_{n+m-2}, p) + e^{\alpha_{n+m-1}+\alpha_{n+m-2}} \cdot [\beta_{n+m-1} \\ &\quad + \beta_{n+m-2}] + e^{\alpha_{n+m-1}+\alpha_{n+m-2}} \cdot [\gamma_{n+m-1} + \gamma_{n+m-1}] \cdot v \\ &\vdots \\ &\preceq N \cdot d(x_n, p) + N \cdot \sum_{i=1}^{n+m-1} \beta_i + N \cdot \Big(\sum_{i=1}^{n+m-1} \gamma_i \Big) \cdot v, \end{aligned}$$

where $N = e^{\sum_{i=1}^{\infty} \alpha_i} > 0$. Hence, (ii) follows from (1.1), since P is a normal cone with the normal constant k. This completes the proof of part (ii). ■

Theorem 4.2. Let K be a nonempty, closed, convex subset of a complete convex cone metric space $X, T : K \to K$ be a generalized asymptotically quasi-nonexpansive mapping with a sequence $\{k_n\}$ and $\{r_n\} \in [0, \infty)$ such that $\sum_{n=1}^{\infty} \{k_n\} < \infty$ and $\sum_{n=1}^{\infty} \{r_n\} < \infty$. Assume that $F = F(T) \neq \phi$. Let $\{x_n\}$ be a SP-iteration process with errors defined by (3.1) and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be bounded sequences in K. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$ and $\{c_n^{'}\}$ be six sequences in $[0, 1]$ with restriction $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1$ and $\sum_{n=1 \to \infty} (c_n + c_n^{'} + c_n^{''}) < \infty$ and if $F = F(T) \neq \phi$. Then, $\{x_n\}$ converges to a common fixed point of S and T if and only if $\liminf_{n \to \infty} \|d(x_n, F)\| = 0$, where $\|d(x, F)\| = \inf\{\|dx, q)\| : q \in F\}$.

Proof. The necessity of condition is obvious. Thus, we will only prove the sufficiency. Then from Lemma 3.8(i), we have

$$\|d(x_{n+1}, p)\| \leq k \cdot (1 + \alpha_n) \cdot \|d(x_n, p)\| + k \cdot \beta_n + k \cdot \|v\| \cdot \gamma_n,$$

where $\alpha_n = 3k_n + 3k_n^2 + k_n^3, \beta_n = (2 + 3k_n + k_n^2) r_n$, $\gamma_n = c_n + c_n^{'} + c_n^{''}$ and $v = \sup_{p \in F, n \geq 1} \{(1+k_n)^2 d(w_n, p) + (1+k_n) d(v_n, p) + d(u_n, p)\}$ with $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ for $(n \in N \cup \{0\})$.

Because $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, it follows from Lemma 3.8 that $\lim_{n\to\infty} \|d(x_n, F)\|$ exists. Now, $\liminf_{n\to\infty} \|d(x_n, F)\| = 0$, therefore, implies that $\lim_{n\to\infty} \|d(x_n, F)\| = 0$. Secondly, we show that $\{x_n\}$ is a Cauchy sequence, for any positive real number ϵ, there exists a natural number $N_0 \in N$ such that $n > N_0$, we get

$$\|d(x_n, F)\| < \frac{\epsilon}{6K^2M} \text{ and } \sum_{n=N_0+1}^{\infty} \beta_n < \frac{\epsilon}{6k^2M},$$

and

$$\sum_{n=N_0+1}^{\infty} \gamma_n < \frac{\epsilon}{6k^2\|v\|M}$$

. In particular, there exists a $q' \in F$ and an integer $n_1 > N_0$ such that

$$\|d(x_1, p_1)\| < \frac{\epsilon}{6k^2M}.$$

It follows from Lemma 3.8 (ii), that when $n > N_1$, we have

$$\begin{aligned}\|d(x_{n+m}, p_1)\| &= \|d(x_{n_1+(n+m-n_1)}, p_1)\| \\ &\leq \|d(x_{n+m}, p)\| \leq k \cdot M \cdot \|d(x_n, p)\| + k \cdot M \cdot \Big(\sum_{i=n_1}^{n+m-1} \beta_i\Big) \\ &\quad + k \cdot M \cdot v \cdot \Big(\sum_{i=n_1}^{n+m-1} \gamma_i\Big), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\|d(x_n, p_1)\| &= \|d(x_{n_1+(n-n_1)}, p_1)\| \\ &\leq \|d(x_{n+m}, p)\| \leq k \cdot M \cdot \|d(x_n, p)\| + k \cdot M \cdot \Big(\sum_{i=n_1}^{n-1} \beta_i\Big) \\ &\quad + k \cdot M \cdot v \cdot \Big(\sum_{i=n_1}^{n-1} \gamma_i\Big). \end{aligned} \tag{4.7}$$

Hence, from (2.1), (4.6) and (4.8), we have

$$\begin{aligned}
\|d(x_{n+m}, x_n)\| &\leq k \cdot \|d(x_{n+m}, p_1) + d(p_1, x_n)\| \\
&\leq k \cdot \|d(x_{n+m}, p_1)\| + k \cdot \|d(p_1, x_n)\| \\
&\leq 2k^2 \cdot M \cdot \|d(x_{n_1}, p_1)\| + 2k^2 \cdot M\Big(\sum_{i=n}^{n+m-1} \beta_i + \sum_{i=n}^{n-1} \beta_i\Big) \\
&\quad + 2k^2 \cdot M \cdot \|v\| \cdot \Big(\sum_{i=n}^{n+m-1} \gamma_i + \sum_{i=n}^{n-1} \gamma_i\Big) \\
&\leq 2k^2 \cdot M \cdot \|d(x_{n_1}, p_1)\| + 2k^2 \cdot M\Big(\sum_{i=n}^{n+m-1} \beta_i\Big) \\
&\quad + 2k^2 \cdot M \cdot \|v\| \cdot \Big(\sum_{i=n}^{n+m-1} \gamma_i\Big) \\
&\leq 2k^2 \cdot M \cdot \frac{\epsilon}{6k^2 M} + 2k^2 \cdot M \cdot \frac{\epsilon}{6k^2 M} + 2k^2 \cdot M \cdot \|v\| \cdot \frac{\epsilon}{6k^2 \|v\| M} \\
&= \epsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in closed convex subset K of a complete cone metric space X. So that it must be convergent to a point in K. Let $\lim_{n\to\infty} x_n = p$. We will prove that $p \in F$.

For given $\epsilon > 0$, there exists an integer n_2 such that for all $n \geq n_2$, we define

$$\|d(x_n, p)\| < \frac{\epsilon}{2k(2 + 3k_1 + k_1^2)} \text{ and } \|d(x_n, F)\| < \frac{\epsilon}{2k(2 + 3k_1 + k_1^2)}. \tag{4.8}$$

In particular, there exists a $p_1 \in F$ and an integer $n_3 > n_2$ such that

$$\|d(x_{n_3}, p)\| < \frac{\epsilon}{2k(2 + 3k_1 + k_1^2)}. \tag{4.9}$$

Then, we obtained

$$\begin{aligned}
d(Tp, p) &\leq d(Tp, p_2) + d(p_2, x_{n_3}) + d(x_{n_3}, p) + 2d(Tx_{n_3}, p_2) \\
&\leq (1 + r)d(p, p_2) + (1 + r)d(x_{n_3}, p_2) + d(x_{n_3}, p_2) \\
&\quad + d(x_{n_3}, p) + 2d(1 + r)(x_{n_3}, p_2)
\end{aligned}$$

Now using (2.1), (4.8) and (4.9), we have

$$\begin{aligned}
d(Tp, p) &\leq k(2 + k_1)d(x_{n_3}, p) + k(2 + k_1)d(x_{n_3}, p_1) \\
&< k(2 + k_1) \cdot \frac{\epsilon}{2k(2 + k_1)} + k(2 + k_1) \cdot \frac{\epsilon}{2k(2 + k_1)} \\
&= \epsilon.
\end{aligned}$$

Similarly, we an also have $\|d(Tp, p)\| < \epsilon$. Since ϵ is arbitrary, that is, p is a common fixed point of S and T. This completes the proof. ■

We can obtain the following result immediately.

Corollary 4.3. Let K be a nonempty, closed, convex subset of a complete convex cone metric space $X, T : K \to K$ be an asymptotically quasi nonexpansive mapping with a sequence $\{k_n\} \in [0, \infty)$. Assume that $F = F(T) \neq \phi$. Let $\{x_n\}$ be the Ishikawa iteration process defined by (1.8) and let $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0, 1] with restriction $\alpha_n + \beta_n = 1$. Then, $\{x_n\}$ converges to a common fixed point of T if and only if $\liminf_{n\to\infty} \|d(x_n, F)\| = 0$, where $\|d(x, F)\| = \inf\{\|dx, q)\| : q \in F\}$.

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