Mathematical model of Unemployment- an analysis with delay

Gulbanu Pathan[1], P.H.Bhathawala[2]

[1]*(Statistical Assistant, Directorate of Economics and Statistics, Gandhinagar, India,*

[2]*(Retd. Prof. & Head, Department of Mathematics, VNSGU, Surat, India,)*

Abstract

This work presented and analyzed a mathematical model for unemployment using four dynamic variables. In this model we analyzed an effect of the action of the government and private sector to control unemployment without any delay and also observe the effect of presence of delay. Also we analyzed the contribution of self-employment for decrease unemployment. We found the non-negative equilibrium point of the system to check the stability. At last Numerical simulation is given to compare with analytical result.

Key words: Employed persons, unemployed persons, self-employment, newly created Vacancies, present jobs, delay.

1. INTRODUCTION:

Unemployment is one of the major problem for the world. Unemployment is not an issue for the particular territory but it is the serious problem for whole world. For a healthy economy of any country high unemployment rate is like a barrier.

Nikolopoulos and Tzanetis ([6]) developed and analyzed a model for a housing allocation of homeless families due to natural disaster. Based on this concept Misra and Singh ([1, 2]) presented a nonlinear mathematical model for unemployment. G.N.Pathan and P.H.Bhathawala ([7]) developed a mathematical model for unemployment with effect of self-employment based on concept of above papers. N.Sirghi, M.Neamtu, and D.Deac presented in ([3]) a nonlinear dynamic model using

four variables: Number of unemployed persons, number of employed persons, number of present jobs in the market and number of newly created vacancies. In ([4]) M. Neamtu presented a model for unemployment based on some concept of ([2, 3]) with adding a new variable number of immigrants. In [9] G.N.Pathan and P.H.Bhathawala developed a mathematical model for unemployment using four dynamic variables without any delay.

Using concept of these papers we developed a dynamic mathematical model for unemployment with four variables: Number of unemployed persons, number of employed persons, number of present jobs in the market and number of newly created vacancies. We introduce an effect of self-employment with the assumption that government and privet sector both tried to create new vacancies without any delay and after that we analyzed the impact of delay in creating new vacancies by government and private sector.

The paper is organized as follows: Section 2 describes Model for unemployment, Section 3 describes an equilibrium analysis, Section 4 describes the stability of equilibrium point, Numerical simulation describes in section 5 and Conclusion is given in section 6.

2. MATHEMATICAL MODEL:

In this process we assume that all entrants of the category unemployment are fully qualified to do any job at any time t. Number of unemployed persons, $U(t)$ increases with constant rate a_1. The rate of movement from unemployed class to employed class is jointly proportional to $U(t)$ and $(P(t) + V(t) - E(t))$. Where $P(t)$ denoted the present jobs in the market available by government and private sector. Government and privet sector try to create new vacancies denoted by $V(t)$ and number of employed persons denoted $E(t)$, Migration as well as death of unemployed persons is proportional to their number with the rate a_3, many times employed person leave the job because of dissatisfaction or fired from their job and joint unemployed class with the rate a_4. Unemployed persons have to create chances for self employment to survive. Unemployed person who start their own independent work and become self employed is proportional to its number with the rate a_5. a_6 is the rate of death and retirement of employed person. The variation in the present job is proportional to c_1 and depreciation rate in present jobs is c_2. α and δ are rate of newly created vacancies and diminution of newly created vacancies. τ describes the delay in creating new vacancies.

$$\frac{dU}{dt} = a_1 - a_2 U(P + V - E) - a_3 U + a_4 E - a_5 U \qquad \_\_\_\_(1)$$

$$\frac{dE}{dt} = a_2 U (P+V-E) - a_4 E + a_5 U - a_6 E \qquad \_\_\_\_(2)$$

$$\frac{dP}{dt} = c_1 U - c_2 P \qquad \_\_\_\_(3)$$

$$\frac{dV}{dt} = \alpha U(t-\tau) - \delta V \qquad \_\_\_\_(4)$$

Lemma 1: The set $\Omega = \{(U,E,P,V) : 0 \le U + E \le \dfrac{a_1}{\gamma}, 0 \le P \le \dfrac{c_1 a_1}{c_2 \gamma}, 0 \le V \le \dfrac{\alpha a_1}{\gamma \delta}\}$,

where $\gamma = \min(a_3, a_6)$ is a region of attraction for the system (1) – (4) and it attracts all solutions initiating in the interior of the positive octant.

Proof:

From equation (1) – (2) we get,

$$\frac{d}{dt}(U(t) + E(t)) = a_1 - a_3 U(t) - a_6 E(t)$$

Which gives

$$\frac{d}{dt}(U(t) + E(t)) \le a_1 - \gamma(U(t) + E(t))$$

Where $\gamma = \min(a_3, a_6)$.

By taking limit supremum

$$\limsup_{t \to \infty} (U(t) + E(t)) \le \frac{a_1}{\gamma}$$

from (3) we have

$$\frac{dP}{dt} = c_1 U(t) - c_2 P(t)$$

$$\therefore \frac{dP}{dt} \le \frac{c_1 a_1}{\gamma} - c_2 P(t)$$

By taking limit supremum which leads to,

$$\limsup_{t\to\infty} P(t) \le \frac{c_1 a_1}{c_2 \gamma}$$

from (4) we have

$$\frac{dV}{dt} = \alpha U(t) - \delta V(t)$$

$$\therefore \frac{dV}{dt} \le \frac{\alpha a_1}{\gamma} - \delta V(t)$$

By taking limit supremum which leads to,

$$\limsup_{t\to\infty} V(t) \le \frac{\alpha a_1}{\delta \gamma}$$

This proves the lemma.

3. EQUILIBRIUM ANALYSIS:

The model system (1) - (4) has only one non negative equilibrium point $E_0(U^*, E^*, P^*, V^*)$ which obtained by solving the following set of algebraic equations.

$$a_1 - a_2 U(P + V - E) - a_3 U + a_4 E - a_5 U = 0 \qquad \underline{\qquad}(5)$$
$$a_2 U(P + V - E) - a_4 E + a_5 U - a_6 E = 0 \qquad \underline{\qquad}(6)$$

$$c_1 U - c_2 P = 0 \qquad \underline{\qquad}(7)$$

$$\alpha U - \delta V = 0 \qquad \underline{\qquad}(8)$$

Taking an addition of equation (5) and (6)

$$a_1 - a_3 U - a_6 E = 0$$
$$\therefore E = \frac{a_1 - a_3 U}{a_6} \qquad \underline{\qquad}(9)$$

From (7)

$$P = \frac{c_1 U}{c_2} \qquad \underline{\qquad}(10)$$

From (8)

$$V = \frac{\alpha U}{\delta} \qquad \underline{\quad}(11)$$

$$\therefore P + V - E = \frac{a a_6 U - a_1}{a_6} \qquad \underline{\quad}(12)$$

Where $a = \frac{\alpha}{\delta} + \frac{a_3}{a_6} + \frac{c_1}{c_2}$

Put values of equation (9) and (12) in (5) we get,

$$A_0 U^2 - A_1 U - A_2 = 0 \qquad \underline{\quad}(13)$$
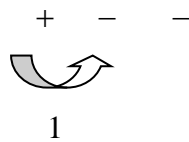
Where,

$$A_0 = a a_2 a_6, \quad A_1 = a_1 a_2 - a_3 (a_4 + a_6) - a_5 a_6,$$

$$A_2 = a_1 (a_4 + a_6).$$

From equation (13)

$$h(U) = A_0 U^2 - A_1 U - A_2 \qquad \underline{\quad}(14)$$

$$+ \quad - \quad -$$

1

Since A_i, *i =0, 1, 2 all* are positive and number of changes in signs of equation (14) is only one. So, by Descart's rule equation (14) has only one positive solution say $U*$. So, we get the non-negative equilibrium point of model with coordinates:

$$E* = \frac{a_1 - a_3 U*}{a_6}$$

$$P* = \frac{c_1 U*}{c_2}$$

$$V* = \frac{\alpha U*}{\delta}$$

So, $E_0 (U*, E*, P*, V*)$ is required non negative solution of the Model.

4.   STABILITY ANALYSIS:

Stability of equilibrium point without any delay:

To check the local stability for $\tau = 0$ at equilibrium point $E_0(U^*,E^*,P^*,V^*)$ we calculate the variational matrix M of the model system (1) – (4) corresponding to $E_0(U^*,E^*,P^*,V^*)$.

$$M = \begin{bmatrix} -q_{11} & q_{12} & -l_2 & -l_2 \\ q_{21} & -q_{22} & l_2 & l_2 \\ c_1 & 0 & -c_2 & 0 \\ \alpha & 0 & 0 & -\delta \end{bmatrix}$$

Where

$l_1 = a_2(P+V-E)$, $l_2 = a_2U$, $q_{11} = l_1 + a_3 + a_5$,

$q_{12} = l_2 + a_4$, $q_{21} = l_1 + a_5$, $q_{22} = l_2 + a_4 + a_6$

The characteristic equation of above matrix is

$$\lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4 = 0 \qquad \underline{\qquad}(15)$$

Where

$d_1 = q_{11} + q_{22} + c_2 + \delta$,

$d_2 = q_{11}(q_{22} + c_2 + \delta) + q_{22}(c_2 + \delta) + c_2\delta - q_{12}q_{21} + c_1l_2 + l_2\alpha$,

$d_3 = q_{11}[q_{22}(c_2 + \delta) + c_2\delta] + c_2q_{22}\delta - q_{12}[q_{21}(c_2 + \delta) + c_1l_2 + l_2\alpha]$

$\qquad + c_1l_2(q_{22} + \delta) + l_2\alpha(q_{22} + c_2)$,

$d_4 = q_{11}c_2q_{22}\delta - q_{12}[q_{21}c_2\delta + l_2(c_1\delta + \alpha c_2)] + c_1q_{22}l_2\delta + l_2q_{22}c_2\alpha$

Since, d_1, d_2, d_3, d_4 are positive then all coefficients of   equation (15) are positive and some algebraic manipulation convey that $d_1d_2 > d_3$ and $d_1d_2d_3 > d_3^2 + d_1^2d_4$. So, by Routh Hurwitz criteria all roots of equation (15) are negative or having a negative real part. Therefore equilibrium point $E_0 = (U^*,E^*,P^*,V^*)$ is locally asymptotically stable.

Stability of equilibrium point with delay:

To check the local stability for $\tau \neq 0$ at equilibrium point $E_0(U^*, E^*, P^*, V^*)$ we calculate the variational matrix M_1 and M_2 of the model system $(1) - (4)$ corresponding to $E_0(U^*, E^*, P^*, V^*)$.

$$\frac{dx}{dt} = M_1 x(t) + M_2 x(t - \tau)$$
_____(16)

Where $x(t) = \begin{bmatrix} u(t) & e(t) & p(t) & v(t) \end{bmatrix}^T$

$u(t)$, $e(t)$, $p(t)$ and $v(t)$ are small perturbations around the equilibrium point E_0

$$M_1 = \begin{bmatrix} -q_{11} & q_{12} & -l_2 & -l_2 \\ q_{21} & -q_{22} & l_2 & l_2 \\ c_1 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & -\delta \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{bmatrix}$$

Where $l_1 = a_2(P + V - E)$, $l_2 = a_2 U$, $q_{11} = l_1 + a_3 + a_5$, $q_{12} = l_2 + a_4$, $q_{21} = l_1 + a_5$,

$$q_{22} = l_2 + a_4 + a_6$$

The characteristic equation of system (16) is

$$\psi^4 + j_1\psi^3 + j_2\psi^2 + j_3\psi + j_4 + (k_1\psi^2 + k_2\psi + k_3)e^{-\psi\tau} = 0$$
_____(17)

Where

$j_1 = q_{11} + q_{22} + c_2 + \delta$,

$j_2 = q_{11}(q_{22} + c_2 + \delta) + q_{22}(c_2 + \delta) + c_2\delta - q_{12}q_{21} + c_1l_2$,

$j_3 = q_{11}[q_{22}(c_2 + \delta) + c_2\delta] + c_2q_{22}\delta - q_{12}[q_{21}(c_2 + \delta) + c_1l_2]$

$\qquad + c_1l_2(q_{22} + \delta)$,

$j_4 = q_{11}c_2q_{22}\delta - q_{12}[q_{21}c_2\delta + l_2c_1\delta] + c_1q_{22}l_2\delta$,

$k_1 = l_2\alpha$, $\quad k_2 = \alpha l_2(q_{22} + c_2 - q_{12})$, $\quad k_3 = \alpha l_2 c_2(q_{22} - q_{12}) = \alpha l_2 c_2 a_6$.

Now to check the stability of Eq. (17) we should not directly use Routh-Hurwitz criterion.

We check that Hopf -bifurcation occurs and for that we have to show that Eq. (17) has a pair of purely imaginary roots. For this we substitute $\psi = i\omega$ in Eq. (17) and we get

$$\omega^4 - j_1\omega^3 i - j_2\omega^2 + j_3\omega i + j_4 + (-k_1\omega^2 + ik_2\omega + k_3)e^{-i\omega\tau} = 0 \qquad _____(18)$$

$$\therefore \omega^4 - j_1\omega^3 i - j_2\omega^2 + j_3\omega i + j_4 + (-k_1\omega^2 + ik_2\omega + k_3)(\cos\omega\tau - i\sin\omega\tau) = 0$$

$$\therefore \omega^4 - j_2\omega^2 + j_4 - (k_1\omega^2 - k_3)\cos\omega\tau + k_2\omega\sin\omega\tau$$
$$+ i(-j_1\omega^3 + j_3\omega - (-k_1\omega^2 + k_3)\sin\omega\tau + k_2\omega\cos\omega\tau) = 0 \qquad _____(19)$$

Separating real and imaginary part of Eq. (19) we get

$$\omega^4 - j_2\omega^2 + j_4 = (k_1\omega^2 - k_3)\cos\omega\tau - k_2\omega\sin\omega\tau \qquad \_\_\_\_\_(20)$$

$$j_1\omega^3 - j_3\omega = (k_1\omega^2 - k_3)\sin\omega\tau + k_2\omega\cos\omega\tau \qquad \_\_\_\_\_(21)$$

By squaring and adding Eq. (20) and Eq. (21)

$$(\omega^4 - j_2\omega^2 + j_4)^2 + (j_1\omega^3 - j_3\omega)^2 = (k_1\omega^2 - k_3)^2 + k_2^2\omega^2 \qquad _____(22)$$

By taking expansion of this

$$\omega^8 + r_1\omega^6 + r_2\omega^4 + r_3\omega^2 + r_4 = 0 \qquad _____(23)$$

$$r_1 = (j_1^2 - 2j_2), \quad r_2 = (j_2^2 + 2j_4 - 2j_1j_3 - k_1^2),$$

$$r_3 = (j_3^2 - 2j_2j_4 + 2k_1k_3 - k_2^2), \quad r_4 = (j_4^2 - k_3^2)$$

Substituting $\omega^2 = \sigma$ in above Eq. then we have

$$f(\sigma) = \sigma^4 + e_1\sigma^3 + e_2\sigma^2 + e_3\sigma + e_4 = 0 \qquad _____(24)$$

Where

$$e_1 = (j_1^2 - 2j_2), \quad e_2 = (j_2^2 + 2j_4 - 2j_1j_3 - k_1^2),$$

$$e_3 = (j_3^2 - 2j_2j_4 + 2k_1k_3 - k_2^2), \quad e_4 = (j_4^2 - k_3^2)$$

If all $e_i > 0$ $(i=1,2,3,4)$ and satisfies Routh-Hurwitz criterion then there is no positive root of Eq. (24) i.e. all roots of Eq. (24) are negative or having a negative real part. So, by Routh-Hurwitz criterion equilibrium E_0 is asymptotically stable for all delay $\tau > 0$.

Contrary if all e_i does not satisfy the Routh-Hurwitz criterion then there is at least one positive root ω_0 exist of Eq. (24) for $e_4 < 0$. From this we get that $(j_4{}^2 - k_3{}^2) < 0$ since $j_4 + k_3 > 0$ so , $j_4 - k_3 < 0$. Which gives the condition for the existence of pair of purely imaginary roots $(\pm i\omega)$ of Eq. (17).

$$j_4 - k_3 < 0$$

$$\therefore c_2\delta\{a_2 a_6(P+V-E)+a_3(a_2 U + a_4 + a_6)+a_5 a_6\}+a_2 a_6 U(c_1\delta - \alpha c_2) < 0 \quad \underline{\quad}(25)$$

From Eq. (20) and Eq. (21) we get

$$\tan\omega\tau = \frac{(k_1\omega^2 - k_3)(j_1\omega^3 - j_3\omega)-k_2\omega(\omega^4 - j_2\omega^2 + j_4)}{k_2\omega(j_1\omega^3 - j_3\omega)+(k_1\omega^2 - k_3)(\omega^4 - j_2\omega^2 + j_4)}$$

For positive ω_0 we have corresponding τ_0 is given by

$$\tau_n = \frac{n\pi}{\omega_0}+\frac{1}{\omega_0}\tan^{-1}(\frac{(k_1\omega_0{}^2 - k_3)(j_1\omega_0{}^3 - j_3\omega_0)-k_2\omega_0(\omega_0{}^4 - j_2\omega_0{}^2 + j_4)}{k_2\omega_0(j_1\omega_0{}^3 - j_3\omega_0)+(k_1\omega_0{}^2 - k_3)(\omega_0{}^4 - j_2\omega_0{}^2 + j_4)}) \quad \underline{\quad}(26)$$

n=0, 1, 2, 3…..

By Butler's lemma we can say that equilibrium E_0 remains stable for $\tau < \tau_0$.

Now to check that Hopf- bifurcation occurs at τ_0 we have to check that τ_0 satisfies the transversality condition.

Lemma 2: Transversality condition is

$$\text{sgn}\left[\frac{d(\text{Re}(\psi))}{d\tau}\right]_{\tau=\tau_0} > 0$$

Proof: By differentiating Eq. (17) with respect to τ, we have

$$(\frac{d\psi}{d\tau})^{-1} = \frac{4\psi^3 + 3j_1\psi^2 + 2j_2\psi + j_3 + (2k_1\psi + k_2)e^{-\psi\tau}}{\psi(k_1\psi^2 + k_2\psi + k_3)e^{-\psi\tau}} - \frac{\tau}{\psi}$$

Now,

$$\text{sgn}\left[\frac{d(\text{Re}(\psi))}{d\tau}\right]_{\tau=\tau_0} = \text{sgn}\left[\frac{d(\text{Re}(\psi))}{d\tau}\right]_{\tau=\tau_0}^{-1}$$

$$= \text{sgn}\left[\text{Re}\left(\frac{d\psi}{d\tau}\right)^{-1}\right]_{\psi=i\omega}$$

$$= \text{sgn}\left[\frac{4\omega_0^8 + 3m_1\omega_0^6 + 2m_2\omega_0^4 + m_3\omega_0^2}{(k_1\omega_0^2 - k_3)^2 + k_2^2\omega_0^2}\right]$$

Here $m_1 = (j_1^2 - 2j_2)$, $m_2 = (2j_4 + j_2^2 + k_1^2 - 2j_1j_3)$,
$m_3 = (j_3^2 + k_2^2 - 2j_2j_4 - 2k_1k_3)$

Since condition (25) is satisfied then we have positive ω_0 and for that Transversally condition is satisfies.

This shows that if condition (25) satisfies then equilibrium E_0 is asymptotically stable for $\tau < \tau_0$ (i.e. $\tau \in [0, \tau_0)$) and   unstable for $\tau > \tau_0$. The condition of Hopf-bifurcation is satisfied so, periodic solution occurs   when τ passes the τ_0 for equilibrium E_0.

5. NUMERICAL SIMULATION:

For the Numerical simulation using MATLAB 7.6.0 we consider the following data,

$a_1 = 5000$, $a_2 = 0.05$, $a_3 = 0.004$, $a_4 = 0.01$, $a_5 = 0.07$, $a_6 = 0.06$,

$c_1 = 0.004$, $c_2 = 0.001$, $\alpha = 0.4$, $\delta = 0.2$,

The equilibrium values of the model are:

$U^* = 13737$, $P^* = 54948$, $E^* = 82417$, $V^* = 27474$.

The eigenvalues of the variational matrix corresponding to the equilibrium point $E_0 = (U^*, P^*, E^*, V^*)$ of model system (1) - (4) (for $\tau = 0$) are: -686.8359, -0.5623, -0.0439 and -0.003. All eigenvalues are negative. So, equilibrium $E_0 = (U^*, P^*, E^*, V^*)$ is locally asymptotically stable.

Using above data, Fig.1 and Fig. 2 represent the graph of variations in the number of unemployed persons with respect to time with difference values of a_2 and a_5 respectively. Fig.1 shows that if rate of unemployed persons to join employed class is increases than number of unemployed person decreases. Fig.2 indicates that if rate of self-employment goes higher than number of unemployed person goes lower but we also observe that for very high rate of self employment we get only limited decrement in unemployment.

Using above all values including values of equilibrium point in Eq. (23) we get $\omega_0 = 0.3535$. Put this value of ω_0 in Eq. (26) we have critical value of τ is $\tau_0 = 5.4873$.
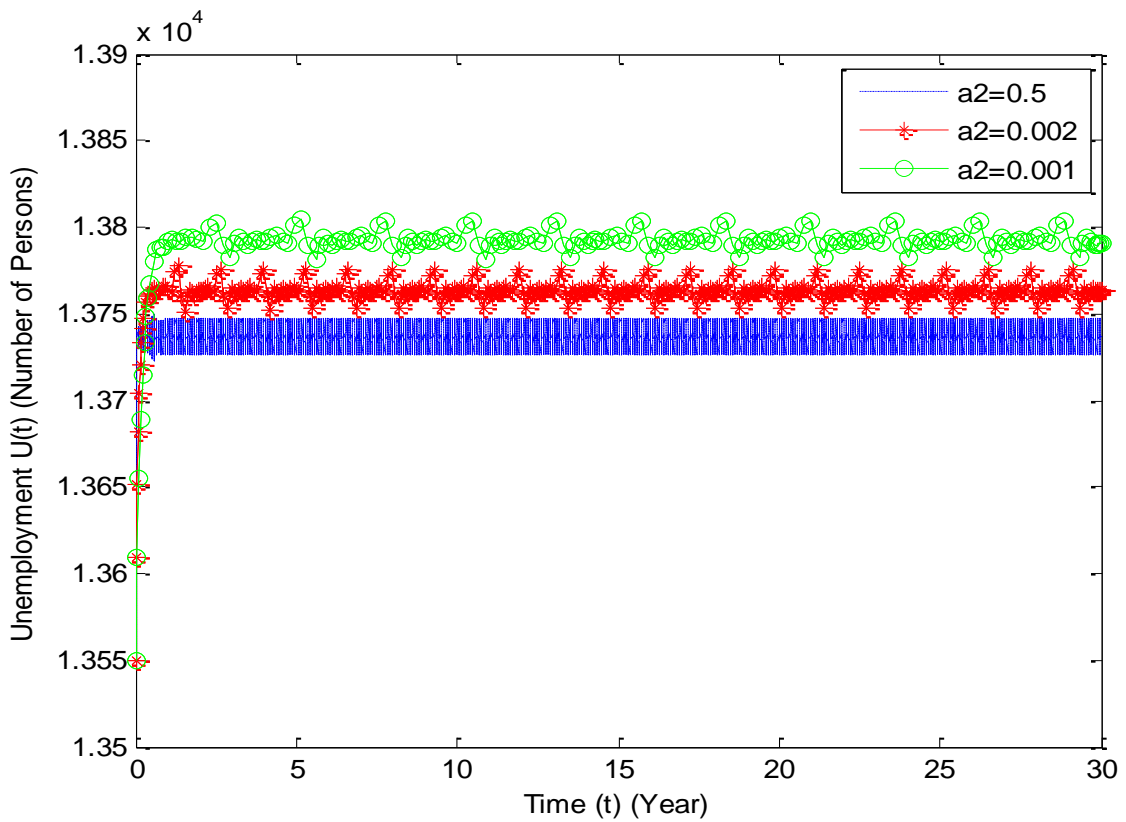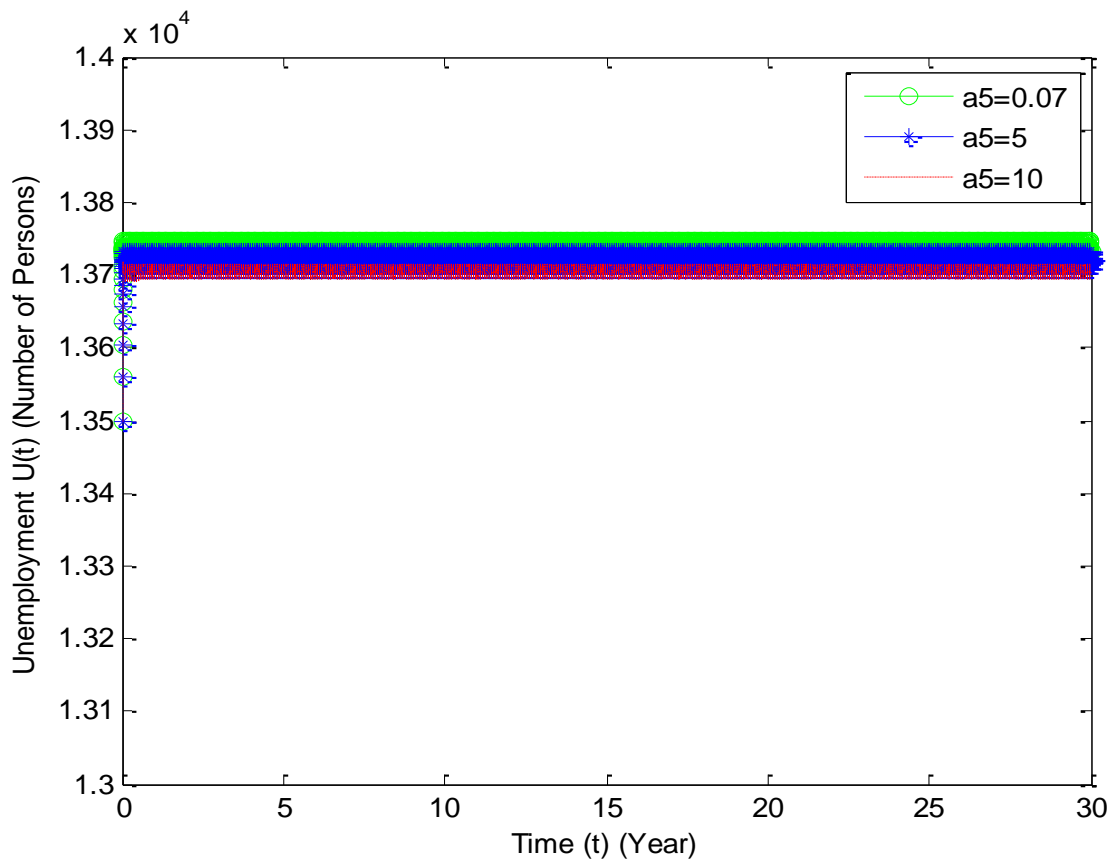
Fig.1

Fig. 2

6. CONCLUSION:

The paper proposed and analyzed a nonlinear mathematical model for unemployment using four dynamic variables: Number of unemployed persons, number of present jobs in the market, number of employed persons and newly created vacancies. We find that equilibrium point is locally asymptotically stable without any condition in absence of delay. But in presence of delay equilibrium point is not always stable. In presence of delay equilibrium point is stable with some condition i.e. if condition (25) satisfies   then we get $\tau_0 > 0$ such that equilibrium point is stable for $\tau < \tau_0$ and unstable for $\tau > \tau_0$.   Theoretical calculation is verified by Numerical simulation which is done   using MATLAB 7.6.0

From above calculations we can see that in absence of delay we get the equilibrium point without any condition. That is unemployment can reduced by improving more newly create vacancies and with higher rate of self-employment. Variation of present jobs also effects the unemployment positively and also negatively according with its increasing or decreasing rate respectively. In presence of delay we get stability with some conditions. It means it is more tough to control unemployment in presence of

delay in compare of absence in delay. From Fig.2 it can be observe that with high self employment rate there is a limited decrement in unemployment. Therefore, to decrease unemployment needs very high self employment rate and also good efforts of government and private sector in creating new vacancies without any delay.

7. REFERENCES:

[1]   A.K.Misra, A.K.Singh, A mathematical model for unemployment, IJTPC 4,2013 pp.2322-3138.

[2]   A.K.Misra, A.K.Singh, A Delay mathematical model for the control of unemployment, Differ Equ Dyn Syst 21(3), 2013,pp.291-307.

[3]   N.Sirghi, M.Neamtu, D.Deac A dynamic model for unemployment control with distributed delay, Mathematical Methods in finance and business Administration, Proceeding of the International Business Administration conference, Tenerife Spain, 1,2014, pp.42-48

[4]   M.Neamtu, A dynamic model for unemployment control with migration and distributed delay, Mathematical Methods in finance and business Administration, Proceeding of the International Business Administration conference, Tenerife, Spain, 1,2014, pp.23-31.

[5]   N. Sirghi, M.Neamtu , Deterministic and stochastic advertising diffusion model With delay, WSEAS Transaction on system and control, 4(8), 2013,pp. 141-150.

[6]   C.V.Nikolopoulos, DE Tzanetis, A model for housing allocation of a homeless population due to natural disaster, Nonlinear Anal.4,2003,pp.561-579.

[7]   G.N.Pathan, P.H.Bhathawala, A Mathematical Model of unemployment with effect of self employment, IOSR- JM, 11(6), 2015, pp.37-43.

[8]   M. Neamtu, M.Pirtea, G. Mircea, D. Opris, Stochastic , Fuzzy, Hybrid delayed Dynamics Heterogeneous competitions with Product Differentiation, WSEAS Transaction on mathematics , 7(9), 2010, pp.581-591.

[9]   G.N.Pathan, P.H.Bhathawala, Unemployment-Discussion with a Mathematical Model, IJBMER, 3(1), 2016,pp.19-27.

[10] G.N.pathan , P.H.Bhathawala, A mathematical model for Unemployment- Taking an action without delay, ADSA, 12(1), 2017, pp.41-48.