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A note on frames for operators in Banach spaces

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Abstract

In this paper, we study frames for bounded linear operators and defined the notion of $\mathcal{A}_d$-operator frame for Banach spaces. A necessary and sufficient condition for a sequence of bounded linear operators to be an $\mathcal{A}_d$-operator frame has been given. Some characterizations of $\mathcal{A}_d$-operator frames have been discussed. Further, a method has been given to generate a Λ-Banach frame using a Schauder frame. In the sequel, an application of this method has been demonstrated.

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1. Introduction

Hilbert frames were formally introduced by Duffin and Scheaffer [8] in 1952 for Hilbert spaces. A natural extension of the Hilbert frame to Banach spaces was introduced by Grochenig in [9], called Banach frame. Since then a number of generalizations of frames in Hilbert and Banach spaces have been appeared in the literature some of them are X_d-frame [2], p-frame [1], G-frame [17], fusion frame [4], fusion Banach frame [12], retro Banach frame [13] etc. The introductory part of Hilbert frames and related concepts can be found in the textbook by Christensen [7] and articles by Casazza [3, 5].

Operator frames for Hilbert spaces were studied by Li and Cao in [15]. In 2012, Chun Yan Li [16] generalized operator frames from Hilbert spaces to Banach spaces. Recently, operator Banach frames in Banach spaces were introduced and studied by Chander Shekhar [6]. During the study of reconstruction property in Banach frame

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theory, Kaushik et al.[14] introduced the concept of Banach Λ-frame by setting the space of bounded linear operators. This notion was further generalized by the present authors in [10] and defined Λ-Banach frames for operator spaces.

In the present paper, we define the notion of associated operator frames for operator spaces and called it $\mathcal{A}_d$-operator frame. Some characterizations of $\mathcal{A}_d$-operator frames have been given which generalizes some results of Casazza et al.[2]. Moreover, we develop a method to construct Λ-Banach frame by Schauder frames. Finally, we illustrate this method by providing an application in the l_1 space.

2. Preliminaries

Throughout the paper, X denote a Banach space and X^* denote a dual space of X. We assume that $\{X_i\}$ be a sequence of Banach spaces over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). The family of bounded linear operators from a Banach space X to a Banach space Y will be denoted by $B(X, Y)$. If $X = Y$, then we write $B(X, Y) = B(X)$. An operator $T \in B(X, Y)$ is said to be coercive if there exists $m > 0$ such that $\|T(x)\| \geq m\|x\|$, for all $x \in X$. The range of T will be denoted by $Ran(T)$.

Definition 2.1. [2] A sequence space X_d is called a BK-space if it is a Banach space and the coordinate functionals are continuous on X_d, i.e. the relations $x_n = \{\alpha_j^{(n)}\}$, $x = \{\alpha_j\} \in X_d$, $\lim_{n\to\infty} x_n = x$ imply that $\lim_{n\to\infty} \alpha_j^{(n)} = \alpha_j$ $(j = 1, 2, \ldots, n)$.

Definition 2.2. [9] Let X be a Banach space and X_d be an associated Banach space of scalar valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset X^*$ and $S : X_d \to X$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame* for X with respect to X_d if

(i) $\{f_n(x)\} \in X_d$, for each $x \in X$,

(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \|\{f_n(x)\}\|_{X_d} \leq B\|x\|_X, \qquad x \in X, \tag{1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \qquad x \in X.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : X_d \to X$ is called the *reconstruction operator* (or, the *pre frame operator*). The inequality (1) is called the *frame inequality*.

Definition 2.3. [6] Let X be a Banach space, $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of Banach spaces and $T_i \in B(X, X_i)$, $i \in \mathbb{N}$. Let $\mathcal{A}$ be an associated Banach space and $S : \mathcal{A} \to X$ be an operator. Then $(\{T_i\}, S)$ is called an *operator Banach frame* (OBF) for X with respect to $\mathcal{A}$ if

(i) $\{T_i f\} \in \mathcal{A}$, $f \in X$,

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_X \leq \|\{T_i f\}\|_{\mathcal{A}} \leq B\|f\|_X, \qquad f \in X. \tag{2}$$

(iii) S is a bounded linear operator such that

$$S(\{T_i f\}) = f, \qquad f \in X.$$

The positive constants A and B, respectively, are called lower and upper frame bounds for the *OBF* $(\{T_i\}, S)$. The inequality (2) is called the frame inequality for the *OBF*. The operator $S : \mathcal{A} \to X$ is called the reconstruction operator.

Definition 2.4. [10] Let X and Y be Banach spaces. Let $\{x_n\}$ be a sequence in X, $\Lambda \in B(X, Y)$ and $S : \mathcal{B}_d \to B(X, Y)$ be an operator, where $\mathcal{B}_d$ be a Banach space of vector valued sequences associated with Y. Then $(\{x_n\}, \Lambda, S)$ is called a Λ-*Banach frame* for $B(X, Y)$ with respect to $\mathcal{B}_d$, if

(i) $\{\Lambda(x_n)\} \in \mathcal{B}_d$, $\Lambda \in B(X, Y)$

(ii) there exist constants $0 < A \leq B < \infty$ such that

$$A\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq B\|\Lambda\|, \qquad \Lambda \in B(X, Y) \tag{3}$$

(iii) S is a bounded linear operator such that

$$S(\{\Lambda(x_n)\}) = \Lambda, \qquad \Lambda \in B(X, Y).$$

Definition 2.5. [11] Let X be a Banach space. A pair $(\{x_n\}, \{f_n\})$ (where $\{x_n\} \in X$, $\{f_n\} \in X^*$) is called a *Schauder frame* for X if

$$x = \sum_{n=1}^{\infty} f_n(x)x_n, \qquad \text{for all } x \in X \tag{4}$$

where the series in (4) converges in the norm topology of X.

3. $\mathcal{A}_d$-operator frame

Let us begin with the following definition of $\mathcal{A}_d$-operator frame.

Definition 3.1. Let X be a Banach space and $\{X_i\}$ be a sequence of Banach spaces. Let X_d be a BK-space. A countable family $\{\Lambda_i\} \subset B(X, X_i)$ is called an $\mathcal{A}_d$-operator frame for X with respect to $X_d = \bigoplus_{i \in \mathbb{N}} X_i$ if

(i) $\{\Lambda_i(x)\} \in X_d, x \in X$.

(ii) there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \|\{\Lambda_i(x)\}\|_{X_d} \leq B\|x\|_X, \qquad x \in X. \tag{5}$$

The positive constants A and B are called lower and upper $\mathcal{A}_d$-frame bounds for X, respectively. The inequality (5) is called $\mathcal{A}_d$-frame inequality. If upper inequality in (5) is satisfied, then $\{\Lambda_i\}$ is called an $\mathcal{A}_d$-Bessel sequence for X with Bessel bound B. The operator $T : X \to X_d$ given by $T(x) = \{\Lambda_i(x)\}$, $x \in X$ is called an *analysis operator*. If there exists a bounded linear operator $S : X_d \to X$ such that $S(\{\Lambda_i(x)\}) = x$, for each $x \in X$, then the system $(\{\Lambda_i\}, S)$ becomes an operator Banach frame for X with respect to X_d. The operator S is called a *pre-frame operator* for $\{\Lambda_i\}$.

Casazza et al. (in Theorem 2.1 [2]) characterized the Banach space X which have an X_d-frame with respect to a given BK-space X_d. Following result generalizes Theorem 2.1 [2] and provides a necessary and sufficient condition for a sequence $\{\Lambda_i\} \subset B(X, X_i)$ to be an $\mathcal{A}_d$-operator frame for X with respect to an associated Banach space X_d.

Theorem 3.2. A sequence of operators $\{\Lambda_i\} \subset B(X, X_i)$ is an $\mathcal{A}_d$-operator frame for X with respect to X_d if and only if X is isomorphic to a closed subspace of X_d.

Proof. Let A and B are the $\mathcal{A}_d$-frame bounds for $\mathcal{A}_d$-operator frame $\{\Lambda_i\}$, then the $\mathcal{A}_d$-frame inequality is given by

$$A\|x\|_X \leq \|\{\Lambda_i(x)\}\|_{X_d} \leq B\|x\|_X, \qquad x \in X. \tag{6}$$

By using lower frame inequality in (6), the analysis operator T of $\{\Lambda_i\}$ is coercive. Thus T is injective and has closed range. From the inverse mapping theorem, X is isomorphic to the range $T(X)$, which is a closed subspace of X_d.

For the reverse part, assume that M is a closed subspace of X_d and U is an isomorphism from X onto M. Let $\{\mathcal{W}_i\}$ be the sequence of coordinate operators on X_d then $\mathcal{W}_i(\{z_j\}_{j\in\mathbb{N}}) = z_i$, for all $i \in \mathbb{N}$.

Choose $\Lambda_i(x) = \mathcal{W}_i U(x)$, $i \in \mathbb{N}$. Then, for all $x \in X$, we have

$$\|x\| = \|U^{-1}U(x)\| \leq \|U^{-1}\|\|Ux\|.$$

So that,

$$\frac{\|x\|}{\|U^{-1}\|} \leq \|\{\Lambda_i(x)\}\| = \|\{\mathcal{W}_i U(x)\}\| = \|Ux\| \leq \|U\|\|x\|, x \in X.$$

That is

$$A\|x\|_X \leq \|\{\Lambda_i(x)\}\|_{X_d} \leq B\|x\|_X, \qquad x \in X,$$

where $A = \dfrac{1}{\|U^{-1}\|}$ and $B = \|U\|$. Hence, $\{\Lambda_i\}$ is an $\mathcal{A}_d$-operator frame for X with respect to $\mathcal{X}$. ■

In Theorem 2.4 [2] Casazza et al., gave a characterization of a Banach space X possessing Banach frame. We now generalize Theorem 2.4 [2] and obtained a necessary and sufficient condition of a Banach space X to possess an *OBF* with respect to X_d.

Theorem 3.3. A system $(\{\Lambda_i\}, S)$ is an OBF for X with respect to X_d if and only if X is isomorphic to a complemented subspace of X_d.

Proof. Let $(\{\Lambda_i\}, S)$ be an *OBF* for X with respect to X_d. Let T and S be analysis operator and pre-frame operator, respectively, for the *OBF* $(\{\Lambda_i\}, S)$. Then $ST = I$ is an identity operator on X. Choose $P = TS$. Then $P^2 = P$ and $Ran(P) = Ran(T)$. Therefore, P is a projection from X_d to the range of T. Thus $T : X \to Ran(T)$ is an isomorphism and $Ran(T)$ is complemented subspace of X_d.

For the reverse part, if $U : X \to M$ is an isomorphism, where M is the complemented subspace of X_d. Then, by Theorem 3.2, $(\{\Lambda_i\}, S)$ is an *OBF* for X with respect to X_d. ■

If $\{x_n\}$ is a Hilbert frame for a Hilbert space H and $V : H \to H$ be an invertible operator, then $\{Vx_n\}$ is a Hilbert frame for H (see Corollary 5.3.2 in [7]). Next, we extend this result to the class $\mathcal{A}_d$-operator frame.

Theorem 3.4. Let $\{\Lambda_i\}$ be an $\mathcal{A}_d$-operator frame for X with respect to X_d and $V \in B(X)$ be an invertible operator. Then $\{\Lambda_i V\}_{i\in\mathbb{N}}$ is an $\mathcal{A}_d$-operator frame for X with $\mathcal{A}_d$-frame bounds $\dfrac{A}{\|V^{-1}\|}$ and $B\|V\|$.

Proof. Let $\{\Lambda_i\}$ be an $\mathcal{A}_d$-operator frame for X with respect to X_d. Then, there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \|\{\Lambda_i(x)\}\|_{X_d} \leq B\|x\|_X, \qquad x \in X.$$

So that we can write,

$$A\|Vx\|_X \leq \|\{\Lambda_i(Vx)\}\|_{X_d} \leq B\|Vx\|_X, \qquad x \in X.$$

Since V is invertible, we obtain

$$A\|V^{-1}\|^{-1}\|x\|_X \leq \|\{\Lambda_i(Vx)\}\|_{X_d} \leq B\|V\|\|x\|_X, \qquad x \in X.$$

Hence, $\{\Lambda_i V\}_{i\in\mathbb{N}}$ is an X_d-operator frame for X with frame bounds $\dfrac{A}{\|V^{-1}\|}$ and $B\|V\|$. ■

Corollary 3.5. Let $\{\Lambda_i\}$ be an $\mathcal{A}_d$-operator frame for X with respect to X_d and $V : X \to X$ be an isometry. Then $\{\Lambda_i V\}_{i\in\mathbb{N}}$ is an $\mathcal{A}_d$-operator frame for X with the same bounds.

Proof. Straight forward. ■

Corollary 3.6. Let $\{\Lambda_i\}$ be an $\mathcal{A}_d$-operator frame for X with respect to X_d and $(I+V) \in B(X)$ be an invertible operator. Then $\{\Lambda_i + \Lambda_i V\}_{i\in\mathbb{N}}$ is an $\mathcal{A}_d$-operator frame for X with $\mathcal{A}_d$-frame bounds $\frac{A}{\|(I+V)^{-1}\|}$ and $B(1 + \|V\|)$.

Proof. Let $\{\Lambda_i\}$ be an $\mathcal{A}_d$-operator frame for X with respect to X_d. Then, there exist constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_X \leq \|\{\Lambda_i(x)\}\|_{X_d} \leq B\|x\|_X, \qquad x \in X.$$

So that we can write,

$$A\|Vx\|_X \leq \|\{\Lambda_i(Vx)\}\|_{X_d} \leq B\|Vx\|_X, \qquad x \in X.$$

Thus, for each $x \in X$, we compute

$$\|\{(\Lambda_i + \Lambda_i V)x\}\|_{X_d} \leq B(1 + \|V\|)\|x\|_X.$$

Again, since $(I + V)$ is invertible, we compute

$$A\|(I+V)^{-1}\|^{-1}\|x\|_X \leq A\|(I+V)(x)\| \leq \|\{(\Lambda_i + \Lambda_i V)(x)\}\|_{X_d}.$$

Thus, the required Λ-frame inequality is

$$A\|(I+V)^{-1}\|^{-1}\|x\|_X \leq \|\{(\Lambda_i + \Lambda_i V)(x)\}\|_{X_d} \leq B(1 + \|V\|)\|x\|_X, \qquad x \in X.$$

Hence, $\{\Lambda_i + \Lambda_i V\}_{i\in\mathbb{N}}$ is an $\mathcal{A}_d$-operator frame for X with frame bounds $\frac{A}{\|(I+V)^{-1}\|}$ and $B(1 + \|V\|)$. ■

Let $\{X_i\}$ be a sequence of Banach spaces. Define for $1 \leq p < \infty$,

$$\oplus_p X_i = \left\{\{x_i\} : x_i \in X_i, i \in \mathbb{N}, \|\{x_i\}\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{\frac{1}{p}} < \infty\right\}$$

$$\text{and } \oplus_\infty X_i = \left\{\{x_i\} : x_i \in X_i, i \in \mathbb{N}, \|\{x_i\}\|_\infty = \sup \|x_i\| < \infty\right\}.$$

Let X, Y be Banach spaces and $\{X_i\}_{i\in\mathbb{N}}$, $\{Y_i\}_{i\in\mathbb{N}}$ be sequences of Banach spaces. Let $\{\Lambda_i\}$ and $\{\Theta_i\}$ be sequences of operators in $B(X, X_i)$ and $B(Y, Y_i)$, respectively. In the next result we shall show that, if $\{\Lambda_i\}$ and $\{\Theta_i\}$ are $\mathcal{A}_d$-operator frames for X and Y respectively, then $\{\Lambda_i \oplus \Theta_i\}$ is an $\mathcal{A}_d$-operator frame for $X \oplus Y$.

Theorem 3.7. Let $\{\Lambda_i\}$ be an $\mathcal{A}_d$-operator frame for X and $\{\Theta_i\}_{i\in\mathbb{N}}$ be an $\mathcal{A}_d$-operator frame for Y. Then $\{\Lambda_i \oplus \Theta_i\}_{i\in\mathbb{N}}$ is an $\mathcal{A}_d$-operator frame for $X \oplus Y$.

Proof. Since $\{\Lambda_i\} \subset B(X, X_i)$ is an $\mathcal{A}_d$-operator frame for X with respect to $X_d = \oplus_{i\in\mathbb{N}} X_i$, there are frame bounds A_1 and B_1 $(0 < A_1 \leq B_1 < \infty)$ satisfying

$$A_1\|x\|_X \leq \|\{\Lambda_i(x)\}\|_{X_d} \leq B_1\|x\|_X, \qquad x \in X. \tag{7}$$

Also, $\{\Theta_i\}_{i\in\mathbb{N}} \subset B(Y, Y_i)$ is an $\mathcal{A}_d$-operator frame for Y with respect to $Y_d = \oplus_{i\in\mathbb{N}} Y_i$, the $\mathcal{A}_d$-frame bounds are given by A_2, B_2 (say). Then the corresponding frame inequality is given by

$$A_2\|y\|_Y \leq \|\{\Theta_i(y)\}\|_{Y_d} \leq B_2\|y\|_Y, \qquad y \in Y. \tag{8}$$

From (7) and (8) we obtain

$$A_1\|x\|_X + A_2\|y\|_X \leq \|\{\Lambda_i(x)\} + \{\Theta_i(y)\}\| \leq B_1\|x\|_X + B_2\|y\|_X. \tag{9}$$

Let $A = \min\{A_1, A_2\}$ and B = $\max\{B_1, B_2\}$. Then we get

$$A\|x \oplus y\|_{X\oplus Y} \leq \|\{(\Lambda_i \oplus \Theta_i)(x \oplus y)\}\|_{Z_d} \leq B\|x \oplus y\|_{X\oplus Y}.$$

Hence, $\{(\Lambda_i \oplus \Theta_i)\}$ is a $\mathcal{A}_d$-operator frame for $X \oplus Y$ with respect to $Z_d = (X_d \oplus Y_d)$. ■

Corollary 3.8. If $\Lambda_i = \{\Lambda_{ij}\}_{j\in\mathbb{N}}$ is an $\mathcal{A}_d$-operator frame for X_i with respect to $\oplus_j X_{ij} = X_d^i$ with $\mathcal{A}_d$-frame bounds A_i and B_i such that $\inf A_i = A > 0$ and $\sup B_i = B < \infty$. Then $\Lambda = \{\oplus_{i\in\mathbb{N}}\Lambda_i\}$ is a $\mathcal{A}_d$-operator frame for $\oplus_{i\in\mathbb{N}} X_i$ with respect to $Z_d = \oplus_i X_d^i$ with $\mathcal{A}_d$-frame bounds A and B.

Similar to Hilbert frames [7], next result shows that the image of an $\mathcal{A}_d$-operator frame under a bounded linear operator is also an $\mathcal{A}_d$-operator frame.

Theorem 3.9. Let $\{\Lambda_i\} \subset B(X, X_i)$ be an $\mathcal{A}_d$-operator frame for X with respect to $X_d = \oplus X_i$ and $S : X \to X$ be a bounded operator. Then $\{\Lambda_i S\}$ is an $\mathcal{A}_d$-operator frame for X with respect to $X_d = \oplus X_i$ if and only if S is bounded below.

Proof. Let $\{\Lambda_i S\}$ be an $\mathcal{A}_d$-operator frame for X with frame bounds m and n. Then we have

$$m\|x\|_X \leq \|\{\Lambda_i S(x)\}\|_{X_d} \leq n\|x\|_X, \qquad x \in X.$$

Let A and B be $\mathcal{A}_d$-frame bounds for $\{\Lambda_i\}$. Then we have

$$A\|Sx\|_X \leq \|\{\Lambda_i(Sx)\}\|_{X_d} \leq B\|Sx\|_X, \qquad x \in X.$$

Thus, we obtain

$$m\|x\|_X \leq B\|Sx\|_X, \qquad x \in X.$$

This shows that $\|Sx\| \geq \delta\|x\|_X$, where $\delta = \dfrac{m}{B} > 0$. Hence, S is bounded below. Further, assume that there exists $\delta > 0$ such that for each $x \in X$, $\|S(x)\| \geq \delta\|x\|_X$. Then, we obtain

$$A\delta\|x\|_X \leq A\|Sx\| \leq \|\{\Lambda_i(Sx)\}\| \leq B\|Sx\| \leq B\|S\|\|x\|_X,$$

therefore, $\{\Lambda_i S\}$ is an $\mathcal{A}_d$-operator frame for X with $\mathcal{A}_d$-frame bounds $A\delta$ and $B\|S\|$. ■

In the next two results, we obtain sufficient conditions for a sequence of operators in $B(X, X_i)$ to be an $\mathcal{A}_d$-operator frame.

Theorem 3.10. Let $\{\Lambda_i\} \subset B(X, X_i)$ be an $\mathcal{A}_d$-operator frame for X, with $\mathcal{A}_d$-frame bounds A and B and let $\{\Theta_i\} \subset B(X, X_i)$ be an $\mathcal{A}_d$-Bessel sequence for X with bound $M < A$, then $\{\Lambda_i \pm \Theta_i\}$ is an $\mathcal{A}_d$-operator frame for X with $\mathcal{A}_d$-frame bounds $(A - M)$ and $(B + M)$.

Proof. Consider the analysis operators $\mathcal{S} : X \to X_d$ and $\mathcal{Q} : X \to X_d$ for $\mathcal{A}_d$-Bessel sequences $\{\Lambda_i\}$ and $\{\Theta_i\}$ are given by $\mathcal{S}(x) = \{\Lambda_i(x)\}$ and $\mathcal{Q}(x) = \{\Theta_i(x)\}$, respectively. Then, for every $x \in X$, we have

$$\begin{aligned}\|\{(\Lambda_i \pm \Theta_i)(x)\}\| =& \|\mathcal{S}(x) \pm \mathcal{Q}(x)\| \\ \leq& \|\{\Lambda_i(x)\}\| + \|\{\Theta_i(x)\}\| \\ \leq& (B + M)\|x\|.\end{aligned}$$

Thus, $\{(\Lambda_i \pm \Theta_i)(x)\}$ is a Bessel sequence for X. We also have for $x \in X$,

$$\begin{aligned}\|\{(\Lambda_i + \Theta_i)(x)\}\| =& \|\mathcal{S}(x) \pm \mathcal{Q}(x)\| \\ \geq& \|\{\Lambda_i(x)\}\| - \|\{\Theta_i(x)\}\| \\ \geq& (A - M)\|x\|.\end{aligned}$$

Hence, $\{\Lambda_i \pm \Theta_i\}$ is an $\mathcal{A}_d$-operator frame for X, with respect to X_d and having desired $\mathcal{A}_d$-frame bounds $(A - M)$ and $(B + M)$. ■

Theorem 3.11. Let $\{\Lambda_i\} \subset B(X, X_i)$ be an $\mathcal{A}_d$-frame for X with frame bounds A and B. Let $\{\Theta_i\} \subset B(X, X_i)$ be such that $\{\Theta_i(x)\} \in X_d$, for all $x \in X$ and let $\{\Lambda_i + \Theta_i\}$ be an $\mathcal{A}_d$-Bessel sequence for X and with bound $M < A$. Then $\{\Theta_i\}$ is an $\mathcal{A}_d$-operator frame for X with bounds $(A - M)$ and $(B + M)$.

Proof. The $\mathcal{A}_d$-frame inequality for the $\mathcal{A}_d$-operator frame $\{\Lambda_i\} \subset B(X, X_i)$ is given by

$$A\|x\| \leq \|\{\Lambda_i(x)\}\|_{X_d} \leq B\|x\|, \qquad x \in X.$$

Since, $\{\Lambda_i + \Theta_i\}$ is an $\mathcal{A}_d$-Bessel sequence for X, we have

$$\|\{(\Lambda_i + \Theta_i)(x)\}\| \leq M\|x\|, \qquad x \in X.$$

Thus we compute,

$$\begin{aligned}(A - M)\|x\| \leq& \|\{\Lambda_i(x)\}\| - \|\{(\Lambda_i + \Theta_i)(x)\}\| \\ \leq& \|\{\Theta_i(x)\}\| \\ \leq& \|\{\Lambda_i(x)\}\| + \|\{(\Lambda_i + \Theta_i)(x)\}\| \\ \leq& (B + M)\|x\|, \qquad x \in X.\end{aligned}$$

Hence, $\{\Theta_i\}$ is an $\mathcal{A}_d$-operator frame for X having desired frame bounds $(A - M)$ and $(B + M)$. ■

4. Λ-Banach frame

Let X and Y be Banach spaces. We now give a characterization of the Schauder frame that the Banach space $B(X, Y)$ of bounded linear operators from X into Y is isomorphic to the Banach space $\mathcal{B}_d$ given by (10) associated with Y. Consequently, we can generate a Λ-Banach frame by Schauder frame as shown in Theorem 3.3 below. Before proceed to the main result we need to prove the following Lemma.

Lemma 4.1. Let X and Y be Banach spaces and let $\{x_n\} \subset X$, $\{f_n\} \subset X^*$ be sequences such that $f_n(x) \neq 0$, $(x \in X, n = 1, 2, \dots)$. Let $\mathcal{B}_d$ be the linear space of sequences of elements

$$\mathcal{B}_d = \left\{ \{z_n\} \subset Y \,\middle|\, \sup_{1 \leq n < \infty} \sup_{\substack{x \in E \\ \|x\| \leq 1}} \left\| \sum_{i=1}^{n} f_i(x) z_i \right\|_Y < \infty \right\} \tag{10}$$

associated with Y and endowed with the norm

$$\|\{z_n\}\|_{\mathcal{B}_d} = \sup_{1 \leq n < \infty} \sup_{\substack{x \in E \\ \|x\| \leq 1}} \left\| \sum_{i=1}^{n} f_i(x) z_i \right\|_Y. \tag{11}$$

Then $\mathcal{B}_d$ is a Banach space.

Proof. If $\|\{z_n\}\|_{\mathcal{B}_d} = 0$, then $\sup_{\substack{x \in E \\ \|x\| \leq 1}} \|f_1(x) z_1\| = 0$. This gives $z_1 = 0$ and hence,

$$\sup_{\substack{x \in E \\ \|x\| \leq 1}} \left\| \sum_{i=1}^{2} f_i(x) z_i \right\| = \sup_{\substack{x \in E \\ \|x\| \leq 1}} \|f_2(x) z_2\| = 0$$

gives $z_2 = 0$. Continuing in this way, we obtain $z_n = 0$ (n = 1,2,...). Hence, norm given in (13) is well defined.

Now we shall show that the space $\mathcal{B}_d$ defined in (10) is a Banach space. Let $\{z_n^{(k)}\}$ $(k = 1, 2, \dots)$ be a Cauchy sequence in $\mathcal{B}_d$. Then for every $\epsilon > 0$ there exists a positive integer n_0 such that

$$\|\{z_n^{(k)}\} - \{z_n^{(m)}\}\| = \sup_{1 \leq n < \infty} \sup_{\substack{x \in E \\ \|x\| \leq 1}} \left\| \sum_{i=1}^{n} f_i(x)(z_i^{(k)} - z_i^{(m)}) \right\|_Y < \epsilon, \qquad (k, m > n_0).$$

Hence,

$$\begin{aligned} \|f_n(x)(z_n^{(k)} - z_n^{(m)})\| &\leq \left\| \sum_{i=1}^{n} f_i(x)(z_i^{(k)} - z_i^{(m)}) \right\| + \left\| \sum_{i=1}^{n-1} f_i(x)(z_i^{(k)} - z_i^{(m)}) \right\| \\ &< 2\epsilon \qquad (k, m > n_0, n = 1, 2, \dots). \end{aligned}$$

Since $f_n(x) \neq 0$, for all $n = 1, 2, \dots,$

$$\|z_n^{(k)} - z_n^{(m)}\| < \frac{2\epsilon}{|f_n(x)|}.$$

Consequently, for each $n \geq 1$ the sequence of vectors $z_n^{(k)}$ $(k = 1, 2, \dots)$ is convergent to a vector z_n. Hence, from the inequalities

$$\left\| \sum_{i=1}^{n} f_i(x)(z_i^{(k)} - z_i^{(m)}) \right\| < \epsilon, \qquad (k, m > n_0; n = 1, 2, \dots).$$

We obtain for $m \to \infty$,

$$\left\| \sum_{i=1}^{n} f_i(x)(z_i^{(k)} - z_i) \right\| \leq \epsilon, \qquad (k > n_0; n = 1, 2, \dots).$$

Then

$$\begin{aligned} \left\| \sum_{i=n+1}^{n+l} f_i(x) z_i \right\| - \left\| \sum_{i=n+1}^{n+l} f_i(x) z_i^{(k)} \right\| &\leq \left\| \sum_{i=n+1}^{n+l} f_i(x)(z_i - z_i^{(k)}) \right\| \\ &= \left\| \sum_{i=1}^{n+l} f_i(x)(z_i - z_i^{(k)}) - \sum_{i=1}^{n} f_i(x)(z_i - z_i^{(k)}) \right\| \\ &\leq \left\| \sum_{i=1}^{n+l} f_i(x)(z_i - z_i^{(k)}) \right\| + \left\| \sum_{i=1}^{n} f_i(x)(z_i - z_i^{(k)}) \right\| \\ &\leq 2\epsilon. \end{aligned}$$

Hence,

$$\left\| \sum_{i=n+1}^{n+l} f_i(x) z_i \right\| \leq 2\epsilon + \left\| \sum_{i=n+1}^{n+l} f_i(x) z_i^{(k)} \right\|.$$

Since, each series $\sum_{i=1}^{\infty} f_i(x) z_i^{(k)}$ is convergent and since Y is complete, it follows that $\sum_{i=1}^{\infty} f_i(x) z_i$ converges, so that

$$\sup_{1 \leq n < \infty} \sup_{\substack{x \in E \\ \|x\| \leq 1}} \left\| \sum_{i=1}^{n} f_i(x) z_i \right\|_Y < \infty.$$

Hence, $\{z_n\} \in \mathcal{B}_d$. Moreover,

$$\|z_n^{(k)} - z_n\| = \sup_{1 \le n < \infty} \sup_{\substack{x \in E \\ \|x\| \le 1}} \left\| \sum_{i=1}^{n} f_i(x)(z_i^{(k)} - z_i) \right\|_Y \le \epsilon, \qquad (k > n_0).$$

Hence, $\mathcal{B}_d$ is a Banach space associated with Y. ∎

We now prove the aforesaid result.

Theorem 4.2. Let X and Y be Banach spaces and let $(\{x_n\}, \{f_n\})$ (where $\{x_n\} \subset X$, $\{f_n\} \subset X^*$) be a Schauder frame for X such that $f_n(x) \neq 0$, $(n = 1, 2, \dots)$. Then the Banach space $B(X, Y)$ is isomorphic, by the mapping $\Lambda \mapsto \{\Lambda(x_n)\}$, to the Banach space of sequences of elements

$$\mathcal{B}_d = \left\{ \{z_n\} \subset Y \,\middle|\, \sup_{1 \le n < \infty} \sup_{\substack{x \in E \\ \|x\| \le 1}} \left\| \sum_{i=1}^{n} f_i(x) z_i \right\|_Y < \infty \right\} \tag{12}$$

associated with Y and endowed with the norm

$$\|\{z_n\}\|_{\mathcal{B}_d} = \sup_{1 \le n < \infty} \sup_{\substack{x \in E \\ \|x\| \le 1}} \left\| \sum_{i=1}^{n} f_i(x) z_i \right\|_Y . \tag{13}$$

Moreover, the system $(\{x_n\}, \Lambda, S)$ is a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$, where $S : \mathcal{B}_d \to B(X, Y)$ be the corresponding Λ-frame operator.

Proof. By Lemma 4.1, $\mathcal{B}_d$ is a Banach space associated with Y. Now, let $\Lambda \in B(X, Y)$ be arbitrary. Put $\Lambda(x_n) = z_n$, $(n = 1, 2, \dots)$ and define $\{\Lambda_n\} \subset B(X, Y)$ by

$$\Lambda_n(x) = \sum_{i=1}^{n} f_i(x) z_i, \qquad (x \in X, n = 1, 2, \dots).$$

Then we have, since $(\{x_n\}, \{f_n\})$ is a Schauder frame,

$$\Lambda(x) = \Lambda\left(\sum_{i=1}^{\infty} f_i(x) x_i \right) = \sum_{i=1}^{\infty} f_i(x) z_i = \lim_{n \to \infty} \sum_{i=1}^{n} f_i(x) z_i = \lim_{n \to \infty} \Lambda_n(x), \qquad (x \in X).$$

Hence, by the principle of uniform boundedness

$$\sup_{1 \le n < \infty} \sup_{\substack{x \in E \\ \|x\| \le 1}} \left\| \sum_{i=1}^{n} f_i(x) z_i \right\|_Y = \sup_{1 \le n < \infty} \|\Lambda_n\| < \infty$$

and thus $\{z_n\} = \{\Lambda(x_n)\} \in \mathcal{B}_d$. Also,

$$\|\Lambda\| \le \sup_{1 \le n < \infty} \|\Lambda_n\| = \sup_{1 \le n < \infty} \sup_{\substack{x \in E \\ \|x\| \le 1}} \left\| \sum_{i=1}^{n} f_i(x) z_i \right\|_Y = \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d}.$$

On the other hand, for every $x \in X$ and $n = 1, 2, \dots,$ we have

$$\Big\| \sum_{i=1}^{n} f_i(x) z_i \Big\|_Y = \Big\| \Lambda\Big(\sum_{i=1}^{n} f_i(x) x_i \Big) \Big\|_Y \leq \|\Lambda\| \Big\| \sum_{i=1}^{n} f_i(x) x_i \Big\|_Y \leq B\|\Lambda\|,$$

where, $B = \| \sum_{i=1}^{n} f_i(x) x_i \|_Y < \infty$. Hence,

$$\|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq B\|\Lambda\|.$$

Since, $\Lambda \in B(X, Y)$ was arbitrary, we obtain the Λ-frame inequality

$$\|\Lambda\| \leq \|\{\Lambda(x_n)\}\|_{\mathcal{B}_d} \leq B\|\Lambda\|, \qquad \Lambda \in B(X, Y).$$

This proves that the mapping $\Lambda \mapsto \{\Lambda(x_n)\}$ is an isomorphism of $B(X, Y)$ into $\mathcal{B}_d$. Define, $S : \mathcal{B}_d \to B(X, Y)$ by $S(\{\Lambda(x_n)\}) = \Lambda$, $\Lambda \in B(X, Y)$. Then S is a bounded linear operator and hence $(\{x_n\}, \Lambda, S)$ is a Λ-Banach frame for X with respect to $\mathcal{B}_d$. ■

Finally, we prove the following result as an application of Theorem 3.3.

Corollary 4.3. Let $X = l_1$ and Y be an arbitrary Banach space. Let $\{x_n\}$ be a sequence of unit vectors in X. Then $B(X, Y)$ is linearly isometric by the mapping $\Lambda \mapsto \{\Lambda(x_n)\}$ to the Banach space of sequences of elements

$$\mathcal{B}_d = \{\{z_n\} \subset F | \sup_{1 \leq n < \infty} \|z_n\| < \infty\} = l_\infty$$

associated with Y and endowed with the norm

$$\|\{z_n\}\|_{\mathcal{B}_d} = \sup_{1 \leq n < \infty} \|z_n\|. \tag{14}$$

Moreover, the system $(\{x_n\}, \Lambda, S)$ is a Λ-Banach frame for $B(X, Y)$ with respect to $\mathcal{B}_d$, where $S : \mathcal{B}_d \to B(X, Y)$ be the corresponding Λ-frame operator.

Proof. Define a sequence $\{f_n\}$ in X^* by

$$f_n(x) = \xi_n, \qquad \text{where } x = \{\xi_n\} \in l^1.$$

Then, we obtain

$$\sum_{i=1}^{\infty} f_n(x) x_n = x.$$

Hence, $(\{x_n\}, \{f_n\})$ is a Schauder frame for X. Also, $\mathcal{B}_d$ is a Banach space with norm given by (14). Further,

$$\|\{z_n\}\| = \sup_{1 \leq n \infty} \sup_{\sum_{j=1}^{\infty} \leq 1} \| \sum_{i=1}^{n} \xi_i z_i \| = \sup_{1 \leq n < \infty} \|z_n\|.$$

Hence, result follows by applying Theorem 3.3. ■

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