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Location of Zeros of Polynomials with Restricted Coefficients

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Abstract

The classical Eneström-Kakeya Theorem is extend to a larger class of polynomials by relaxing the hypothesis in different ways which in turn many other results, and some exetention of Eneström-Kakeya Theorem.

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1. Introduction

Finding the roots of a polynomial is a long standing classical problem [5, 6]. The various results in the analytic theory of polynomials concerning the number of zeros in a region have been frequently investigated. It is an interesting area of research for engineers as well as mathematicians and many results on the same topic are available in the literature. Over last five decades, a large number of research papers, e.g, [3, 4, 7–10] have been published.

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Polynomials in various forms have recently come under extensive revision because of their applications in linear control systems, signal processing, electrical networks, coding theory and several areas of physical sciences, where among others location of zeros and stability problems arise in a natural way. Existing results in the literature also show that there is need to find bounds for special polynomials, for example, those having restrictions on the coefficient and there is always need for refinement of results in this subject. Among them the Eneström-Kakeya theorem [1, 2] is a very strong tool to find the region in the complex plane containing all the zeros of a class of polynomials. It has been used to analyze overflow oscillation of discrete-time dynamical system [11], to investigate the properties of orthogonal wavelets [12].

The following result is well known in the theory of the distribution of zeros of polynomials.

Theorem 1.1. [Eneström-Kakeya theorem] Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that

$$0 < a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq a_n$$

then all the zeros of P(z) lie in $|z| \leq 1$.

Here we prove more generalized results by using Eneström-Kakeya Theorem.

Theorem 1.2. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that for some $k \geq 1, \delta > 0$,

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1}$$
$$\geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if both n and $(n-m)$ are even or odd
(OR)

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \ldots \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1}$$
$$\geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z+k-1| \leq \frac{1}{|a_n|}\Big\{2\delta + ka_n + |a_0| - a_0 + 2\big([a_{n-2} + a_{n-4} + \ldots + a_{n-m+2} + a_{n-m}]$$
$$- [a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\Big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z+k-1| \leq \frac{1}{|a_n|}\Big\{2\delta + ka_n + |a_0| - a_0 + 2\big([a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}]$$
$$- [a_{n-1} + a_{n-3} + \cdots + a_{n-m+4} + a_{n-m+2}]\big)\Big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Corollary 1.3. Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \\ \geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0$$

if both n and $(n-m)$ are even or odd
(OR)

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \\ \geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\Big\{|a_0| - a_0 + a_n + 2\big([a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}] \\ - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\Big\}$$

if both n and $(n-m)$ are even or odd
(OR)

$$|z| \leq \frac{1}{|a_n|}\Big\{|a_0| - a_0 + a_n + 2\big([a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}] \\ - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+4} + a_{n-m+2}]\big)\Big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Corollary 1.4. Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that for some $k \geq 1$, $\delta > 0$,

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \\ \geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if both n and $(n-m)$ are even or odd
(OR)

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \\ \geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z+k-1| \leq \frac{1}{|a_n|}\Big\{2\delta + ka_n + 2\big([a_{n-2}+a_{n-4}+\cdots+a_{n-m+2}+a_{n-m}] - [a_{n-1}+a_{n-3}+\cdots+a_{n-m+3}+a_{n-m+1}]\big)\Big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z+k-1| \leq \frac{1}{|a_n|}\Big\{2\delta + ka_n + 2\big([a_{n-2}+a_{n-4}+\cdots+a_{n-m+3}+a_{n-m+1}] - [a_{n-1}+a_{n-3}+\cdots+a_{n-m+4}+a_{n-m+2}]\big)\Big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Remark 1.5.

1. By taking $k=1, \delta=0$ in Theoem 1.2, then it is reduces Corollary 1.3.
2. By taking $a_i > 0$, for $i = 0, 1, 2, \ldots, n$ in Theorem 1.2,then it reduces to Corollary 1.4.

Theorem 1.6. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that for some $0 < r \leq 1, \delta > 0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if both n and $(n-m)$ are even or odd
(OR)

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\Big\{|a_0| - a_0 - a_n + 2\big([a_{n-1}+a_{n-3}+\cdots+a_{n-m+3}+a_{n-m+1}] - [a_{n-2}+a_{n-4}+\cdots+a_{n-m+4}+a_{n-m+2}]\big)\Big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z| \leq \frac{1}{|a_n|}\big\{|a_0| - a_0 - a_n + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+2} + a_{n-m}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Corollary 1.7. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that

$$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0$$

if both n and $(n-m)$ are even or odd
(OR)

$$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\big\{|a_0| - a_0 - a_n + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}]\big)\big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z| \leq \frac{1}{|a_n|}\big\{|a_0| - a_0 - a_n + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+2} + a_{n-m}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Corollary 1.8. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that for some $0 < r \leq 1, \delta > 0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq \cdots \geq a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_0 - \delta$$

if both n and $(n-m)$ are even or odd
(OR)

$$ra_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \ge a_{n-m+1} \le a_{n-m} \ge a_{n-m-1} \\ \ge \cdots \ge a_4 \ge a_3 \ge a_2 \ge a_1 \ge a_0 - \delta$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|a_n|}\big\{2\big([a_{n-1}+a_{n-3}+\cdots+a_{n-m+3}+a_{n-m+1}] \\ -[a_{n-2}+a_{n-4}+\cdots+a_{n-m+4}+a_{n-m+2}]-a_n\big)\big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z| \le \frac{1}{|a_n|}\big\{2\big([a_{n-1}+a_{n-3}+\cdots+a_{n-m+2}+a_{n-m}] \\ -[a_{n-2}+a_{n-4}+\cdots+a_{n-m+3}+a_{n-m+1}]-a_n\big)\big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Remark 1.9.

1. By taking $r=1, \delta=0$ in Theoem 1.6, then it is reduces Corollary 1.7.
2. By taking $a_i > 0$, for $i = 0, 1, 2, \ldots, n$ in Theorem 1.2, then it reduces to Corollary 1.8.

Theorem 1.10. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that for some $k \ge 1, \delta > 0$,

$$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge \cdots \ge a_{n-m+1} \le a_{n-m} \le a_{n-m-1} \\ \le \cdots \le a_4 \le a_3 \le a_2 \le a_1 \le a_0 + \delta$$

if both n and $(n-m)$ are even or odd
(OR)

$$ka_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge \cdots \le a_{n-m+1} \ge a_{n-m} \le a_{n-m-1} \\ \le \cdots \le a_4 \le a_3 \le a_2 \le a_1 \le a_0 + \delta$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z+k-1| \le \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| + a_0 + 2\big([a_{n-2}+a_{n-4}+\cdots+a_{n-m+4}+a_{n-m+2}] \\ -[a_{n-1}+a_{n-3}+\cdots+a_{n-m+3}+a_{n-m+1}]\big)\big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z+k-1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| + a_0 + 2\big([a_{n-2}+a_{n-4}+\cdots+a_{n-m+3}+a_{n-m+1}] - [a_{n-1}+a_{n-3}+\cdots+a_{n-m+2}+a_{n-m}]\big)\big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Corollary 1.11. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq \cdots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0$$

if both n and $(n-m)$ are even or odd
(OR)

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq \cdots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\big\{|a_0| + a_0 + a_n + 2\big([a_{n-2}+a_{n-4}+\cdots+a_{n-m+4}+a_{n-m+2}] - [a_{n-1}+a_{n-3}+\cdots+a_{n-m+3}+a_{n-m+1}]\big)\big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z| \leq \frac{1}{|a_n|}\big\{|a_0| + a_0 + a_n + 2\big([a_{n-2}+a_{n-4}+\cdots+a_{n-m+3}+a_{n-m+1}] - [a_{n-1}+a_{n-3}+\cdots+a_{n-m+2}+a_{n-m}]\big)\big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Corollary 1.12. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that for some $k \geq 1$, $\delta > 0$,

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq \cdots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if both n and $(n-m)$ are even or odd
(OR)

$$ka_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \\ \leq \cdots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z+k-1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + 2\big(a_0 + [a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}] \\ - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z+k-1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + 2\big(a_0 + [a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}] \\ - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+2} + a_{n-m}]\big)\big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Remark 1.13.

1. By taking $k = 1, \delta = 0$ in Theoem 1.10, then it is reduces Corollary 1.11.
2. By taking $a_i > 0$, for $i = 0, 1, 2, \ldots, n$ in Theorem 1.10, then it reduces to Corollary 1.12.

Theorem 1.14. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that for some $0 < r \leq 1, \delta > 0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \\ \leq \cdots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if both n and $(n-m)$ are even or odd
(OR)

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \\ \leq \cdots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\big\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta \\ + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}] \\ - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}]\big)\big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z| \leq \frac{1}{|a_n|}\big\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+4} + a_{n-m+2}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Corollary 1.15. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that

$$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq \cdots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0$$

if both n and $(n-m)$ are even or odd
(OR)

$$a_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \geq a_{n-m+1} \leq a_{n-m} \leq a_{n-m-1} \leq \cdots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\big\{|a_0| + a_0 - a_n + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}]\big)\big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z| \leq \frac{1}{|a_n|}\big\{|a_0| + a_0 - a_n + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+4} + a_{n-m+2}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Corollary 1.16. Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that for some $0 < r \leq 1$, $\delta > 0$,

$$ra_n \leq a_{n-1} \geq a_{n-2} \leq a_{n-3} \geq a_{n-4} \leq \cdots \leq a_{n-m+1} \geq a_{n-m} \leq a_{n-m-1} \leq \cdots \leq a_4 \leq a_3 \leq a_2 \leq a_1 \leq a_0 + \delta$$

if both n and $(n-m)$ are even or odd
(OR)

$$r a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \ge a_{n-m+1} \le a_{n-m} \le a_{n-m-1} \\ \le \cdots \le a_4 \le a_3 \le a_2 \le a_1 \le a_0 + \delta$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even
then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|a_n|}\Big\{(1-2r)a_n + 2\delta + 2\big(a_0 + [a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}] \\ - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}]\big)\Big\}$$

if both n and $(n-m)$ are even or odd (OR)

$$|z| \le \frac{1}{|a_n|}\Big\{(1-2r)a_n + 2\delta + 2\big(a_0 + [a_{n-1} + a_{n-3} + \cdots + a_{n-m+4} + a_{n-m+2}] \\ - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\Big\}$$

if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even.

Remark 1.17.

1. By taking $r = 1, \delta = 0$ in Theoem 1.14, then it is reduces Corollary 1.15.
2. By taking $a_i > 0$, for $i = 0, 1, 2, \ldots, n$ in Theorem 1.14, then it reduces to Corollary 1.16.

2. Proof of the Theorems

Proof of Theorem 1.2.

Let

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots + a_{n-m}z^{n-m} \\ + \cdots + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

be a polynomial of degree $n \ge 2$.

Then consider the polynomial $Q(z) = (1-z)P(z)$ so that

$$Q(z) = -a_n z^n(z + k - 1) + (ka_n - a_{n-1})z^n \\ + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{n-m+1} - a_{n-m})z^{n-m+1} \\ + (a_{n-m} - a_{n-m-1})z^{n-m} + (a_{n-m-1} - a_{n-m-2})z^{n-m-1} \\ + \cdots + (a_3 - a_2)z^3 + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.$$

Also if $|z| > 1$ then $\dfrac{1}{|z|^{n-i}} < 1$ for $i = 1, 2, \dots, n-1$. Now

$$\begin{aligned}
|Q(z)| &\geq |a_n||z+k-1|^n - \big\{|ka_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} \\
&\quad + \cdots + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} \\
&\quad + |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} \\
&\quad + \cdots + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0|z + |a_0|\big\} \\
&\geq |a_n||z|^n\Bigg[|z+k-1| - \frac{1}{|a_n|}\Big\{k|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} \\
&\quad + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + \frac{|a_{n-3} - a_{n-4}|}{|z|^3} \\
&\quad + \cdots + \frac{|a_{n-m+1} - a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m} - a_{n-m-1}|}{|z|^m} + \frac{|a_{n-m-1} - a_{n-m-2}|}{|z|^{m+1}} \\
&\quad + \cdots + \frac{|a_3 - a_2|}{|z|^{n-3}} + \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Big\}\Bigg] \\
&\geq |a_n||z|^n\Bigg[|z+k-1| - \frac{1}{|a_n|}\Big\{k|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| \\
&\quad + |a_{n-2} - a_{n-3}| + |a_{n-3} - a_{n-4}| \\
&\quad + \cdots + |a_{n-m+1} - a_{n-m}| + |a_{n-m} - a_{n-m-1}| + |a_{n-m-1} - a_{n-m-2}| \\
&\quad + \cdots + |a_3 - a_2| + |a_2 - a_1| + |a_1 - \delta - a_0 + \delta| + |a_0|\Big\}\Bigg] \\
&\geq |a_n||z|^n\Bigg[|z+k-1| - \frac{1}{|a_n|}\Big\{(ka_n - a_{n-1}) + (a_{n-2} - a_{n-1}) \\
&\quad + (a_{n-2} - a_{n-3}) + (a_{n-4} - a_{n-3}) \\
&\quad + \cdots + (a_{n-m} - a_{n-m+1}) + (a_{n-m} - a_{n-m-1}) + (a_{n-m-1} - a_{n-m-2}) \\
&\quad + \cdots + (a_3 - a_2) + (a_2 - a_1) + (a_1 + \delta - a_0) + \delta + |a_0|\Big\}\Bigg]
\end{aligned}$$

if both n and $(n-m)$ are even or odd

$$\begin{aligned}
&= |a_n||z|^n\Bigg[|z+k-1| - \frac{1}{|a_n|}\Big\{2\delta + ka_n + |a_0| - a_0 \\
&\quad + 2\big([a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}] \\
&\quad - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\Big\}\Bigg] \\
&> 0
\end{aligned}$$

if

$$|z+k-1| > \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| - a_0 \\ + 2\big([a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}] \\ - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\big\}$$

This shows that if $|z| > 1$, then $Q(z) > 0$,
if

$$|z+k-1| > \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| - a_0 \\ + 2\big([a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}] \\ - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\big\}.$$

Hence all the zeros of $Q(z)$ with $|z| > 1$ lie in

$$|z+k-1| \le \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| - a_0 \\ + 2\big([a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}] \\ - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\big\}.$$

if both n and $(n-m)$ even or odd.

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem 1.2, if both n and $(n-m)$ even or odd.

Similarly we can also prove for n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even degree polynomials. For this we can rearrange the terms. That is if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even then all the zeros $P(z)$ lie in

$$|z+k-1| \le \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| - a_0 + 2\big([a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}] \\ - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+4} + a_{n-m+2}]\big)\big\}.$$

■

Proof of Theorem 1.6.

Let

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} \\ + \cdots + a_{n-m}z^{n-m} + \cdots + a_3z^3 + a_2z^2 + a_1z + a_0$$

be a polynomial of degree $n \ge 2$.

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$$\begin{aligned} Q(z) = &-a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} \\ &+ \cdots + (a_{n-m+1} - a_{n-m})z^{n-m+1} \\ &+ (a_{n-m} - a_{n-m-1})z^{n-m} + (a_{n-m-1} - a_{n-m-2})z^{n-m-1} \\ &+ \cdots + (a_3 - a_2)z^3 + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0 \end{aligned}$$

Also if $|z| > 1$ then $\dfrac{1}{|z|^{n-i}} < 1$ for $i = 0, 1, 2, \ldots, n-1$. Now

$$\begin{aligned} |Q(z)| \geq\ & |a_n||z|^{n+1} - \big\{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} \\ &+ \cdots + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} \\ &+ |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} \\ &+ \cdots + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0|z + |a_0|\big\} \\ \geq\ & |a_n||z|^n\bigg[|z| - \frac{1}{|a_n|}\big\{|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} \\ &+ \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + \frac{|a_{n-3} - a_{n-4}|}{|z|^3} + \cdots \\ &+ \frac{|a_{n-m+1} - a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m} - a_{n-m-1}|}{|z|^m} \\ &+ \frac{|a_{n-m-1} - a_{n-m-2}|}{|z|^{m+1}} \\ &+ \cdots + \frac{|a_3 - a_2|}{|z|^{n-3}} + \frac{|a_2 - a_1|}{|z|^{n-2}} \\ &+ \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\big\}\bigg] \\ \geq\ & |a_n||z|^n\bigg[|z| - \frac{1}{|a_n|}\{|ra_n - a_{n-1} - ra_n + a_n| + |a_{n-1} - a_{n-2}| \\ &+ |a_{n-2} - a_{n-3}| + |a_{n-3} - a_{n-4}| \\ &+ \cdots + |a_{n-m+1} - a_{n-m}| + |a_{n-m} - a_{n-m-1}| + |a_{n-m-1} - a_{n-m-2}| \\ &+ \cdots + |a_3 - a_2| + |a_2 - a_1| + |a_1 + \delta - a_0 - \delta| + |a_0|\big\}\bigg] \end{aligned}$$

$$\geq |a_n||z|^n\Big[|z| - \frac{1}{|a_n|}\{(a_{n-1} - ra_n) + (1-r)|a_n| + (a_{n-1} - a_{n-2})$$
$$+ (a_{n-3} - a_{n-2}) + (a_{n-3} - a_{n-4})$$
$$+ \cdots + (a_{n-m+1} - a_{n-m+2}) + (a_{n-m+1} - a_{n-m})$$
$$+ (a_{n-m} - a_{n-m-1}) + (a_{n-m-1} - a_{n-m-2})$$
$$+ \cdots + (a_3 - a_2) + (a_2 - a_1) + (a_1 - a_0 + \delta) + \delta + |a_0|\}\Big]$$

if both n and $(n-m)$ are even or odd

$$= |a_n||z|^n\Big[|z| - \frac{1}{|a_n|}\big\{|a_n| + |a_0| - a_0 - r(|a_n| + a_n) + 2\delta$$
$$+ 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]$$
$$- [a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}]\big)\big\}\Big]$$
$$> 0$$

if

$$|z| > \frac{1}{|a_n|}\big\{|a_n| + |a_0| - a_0 - r(|a_n| + a_n) + 2\delta$$
$$+ 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]$$
$$- [a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}]\big)\big\}.$$

This shows that if $|z| > 1$, then $Q(z) > 0$, if

$$|z| > \frac{1}{|a_n|}\{|a_n| + |a_0| - a_0 - r(|a_n| + a_n) + 2\delta$$
$$+ 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]$$
$$- [a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}]\big)\big\}.$$

Hence all the zeros of $Q(z)$ with $|z| > 1$ lie in

$$|z| \leq \frac{1}{|a_n|}\big\{|a_n| + |a_0| - a_0 - r(|a_n| + a_n) + 2\delta$$
$$+ 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}]$$
$$- [a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}]\big)\big\},$$

if both n and $(n-m)$ even or odd.

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem 1.6, if both n and $(n-m)$ are even or odd.

Similarly we can also prove for n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even degree polynomials. For this we can rearrange the terms. That is if n is even and $(n-m)$ is odd (or) if n is odd and $(n-m)$ is even then all the zeros $P(z)$ lie in

$$\begin{aligned}|z| \le \frac{1}{|a_n|}\{&|a_n| + |a_0| - a_0 - r(|a_n| + a_n) + 2\delta \\ &+ 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+2} + a_{n-m}] \\ &- [a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\}.\end{aligned}$$

■

Proof of Theorem 1.10.

Let

$$\begin{aligned}P(z) = {}& a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + a_{n-3} z^{n-3} + \cdots + a_{n-m} z^{n-m} \\ &+ \cdots + a_3 z^3 + a_2 z^2 + a_1 z + a_0\end{aligned}$$

be a polynomial of degree $n \ge 2$.

Then consider the polynomial $Q(z) = (1-z)P(z)$ so that

$$\begin{aligned}Q(z) = {}& -a_n z^n (z + k - 1) + (k a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} \\ &+ \cdots + (a_{n-m+1} - a_{n-m}) z^{n-m+1} \\ &+ (a_{n-m} - a_{n-m-1}) z^{n-m} + (a_{n-m-1} - a_{n-m-2}) z^{n-m-1} \\ &+ \cdots + (a_3 - a_2) z^3 + (a_2 - a_1) z^2 + (a_1 - a_0) z + a_0\end{aligned}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < 1$ for $i = 0, 1, 2, \ldots, n-1$. Now

$$\begin{aligned}|Q(z)| \ge {}& |a_n||z + k - 1|^n - \{|k a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} \\ &+ \cdots + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} \\ &+ |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} \\ &+ \cdots + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0|z + |a_0|\} \\ \ge {}& |a_n||z|^n \Bigg[|z + k - 1| - \frac{1}{|a_n|}\Big\{k|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} \\ &+ \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + \frac{|a_{n-3} - a_{n-4}|}{|z|^3} + \cdots\end{aligned}$$

$$
\begin{aligned}
&+\frac{|a_{n-m+1}-a_{n-m}|}{|z|^{m-1}}+\frac{|a_{n-m}-a_{n-m-1}|}{|z|^{m}}+\frac{|a_{n-m-1}-a_{n-m-2}|}{|z|^{m+1}}+\cdots\\
&+\frac{|a_3-a_2|}{|z|^{n-3}}+\frac{|a_2-a_1|}{|z|^{n-2}}+\frac{|a_1-a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^{n}}\Big\}\Bigg]\\
&\geq |a_n||z|^n\Bigg[|z+k-1|-\frac{1}{|a_n|}\Big\{k|a_n-a_{n-1}|+|a_{n-1}-a_{n-2}|\\
&+|a_{n-2}-a_{n-3}|+|a_{n-3}-a_{n-4}|+\cdots\\
&+|a_{n-m+1}-a_{n-m}|+|a_{n-m}-a_{n-m-1}|+|a_{n-m-1}-a_{n-m-2}|\\
&+\cdots+|a_3-a_2|+|a_2-a_1|+|a_1-\delta-a_0+\delta|+|a_0|\Big\}\Bigg]\\
&\geq |a_n||z|^n\Bigg[|z+k-1|-\frac{1}{|a_n|}\Big\{(ka_n-a_{n-1})+(a_{n-2}-a_{n-1})\\
&+(a_{n-2}-a_{n-3})+(a_{n-4}-a_{n-3})\\
&+\cdots+(a_{n-m+2}-a_{n-m+1})+(a_{n-m}-a_{n-m+1})\\
&+(a_{n-m-1}-a_{n-m})+(a_{n-m-2}-a_{n-m-1})\\
&+\cdots+(a_2-a_3)+(a_1-a_2)+(a_0+\delta-a_1)+\delta+|a_0|\Big\}\Bigg]
\end{aligned}
$$

if both n and $(n-m)$ are even or odd

$$
\begin{aligned}
&= |a_n||z|^n\Bigg[|z+k-1|-\frac{1}{|a_n|}\Big\{2\delta+ka_n+|a_0|+a_0\\
&\quad+2\big([a_{n-2}+a_{n-4}+\cdots+a_{n-m+4}+a_{n-m+2}]\\
&\quad-[a_{n-1}+a_{n-3}+\cdots+a_{n-m+3}+a_{n-m+1}]\big)\Big\}\Bigg]\\
&> 0
\end{aligned}
$$

if

$$
\begin{aligned}
|z+k-1| > \frac{1}{|a_n|}\Big\{&2\delta+ka_n+|a_0|+a_0\\
&+2\big([a_{n-2}+a_{n-4}+\cdots+a_{n-m+4}+a_{n-m+2}]\\
&-[a_{n-1}+a_{n-3}+\cdots+a_{n-m+3}+a_{n-m+1}]\big)\Big\}.
\end{aligned}
$$

This shows that if $|z|>1$, then $Q(z)>0$, if

$$
\begin{aligned}
|z+k-1| > \frac{1}{|a_n|}\Big\{&2\delta+ka_n+|a_0|-a_0\\
&+2([a_{n-2}+a_{n-4}+\cdots+a_{n-m+4}+a_{n-m+2}]\\
&-[a_{n-1}+a_{n-3}+\cdots+a_{n-m+3}+a_{n-m+1}]\big)\Big\}.
\end{aligned}
$$

Hence all the zeros of $Q(z)$ with $|z| > 1$ lie in

$$|z + k - 1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| + a_0 + 2([a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}] - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}])\big\},$$

if both n and $(n - m)$ even or odd.

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem 1.10, if both n and $(n - m)$ are even or odd.

Similarly we can also prove for n is even and $(n - m)$ is odd (or) if n is odd and $(n - m)$ is even degree polynomials. For this we can rearrange the terms. That is if n is even and $(n - m)$ is odd (or) if n is odd and $(n - m)$ is even then all the zeros $P(z)$ lie in

$$|z + k - 1| \leq \frac{1}{|a_n|}\big\{2\delta + ka_n + |a_0| + a_0 + 2\big([a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}] - [a_{n-1} + a_{n-3} + \cdots + a_{n-m+2} + a_{n-m}]\big)\big\}.$$

■

Proof of Theorem 1.14.

Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + a_{n-3} z^{n-3} + \cdots + a_{n-m} z^{n-m} + \cdots + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

be a polynomial of degree $n \geq 2$.

Then consider the polynomial $Q(z) = (1 - z)P(z)$ so that

$$Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{n-m+1} - a_{n-m})z^{n-m+1} + (a_{n-m} - a_{n-m-1})z^{n-m} + (a_{n-m-1} - a_{n-m-2})z^{n-m-1} + \cdots + (a_3 - a_2)z^3 + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < 1$ for $i = 0, 1, 2, \ldots, n-1$. Now

$$
\begin{aligned}
|Q(z)| \geq{}& |a_n||z|^{n+1} - \big\{|a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} \\
&+ \cdots + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} \\
&+ |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} \\
&+ \cdots + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0|z + |a_0|\big\} \\
\geq{}& |a_n||z|^n\Big[|z| - \frac{1}{|a_n|}\big\{|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} \\
&+ \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + \frac{|a_{n-3} - a_{n-4}|}{|z|^3} \\
&+ \cdots + \frac{|a_{n-m+1} - a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m} - a_{n-m-1}|}{|z|^m} \\
&+ \frac{|a_{n-m-1} - a_{n-m-2}|}{|z|^{m+1}} \\
&+ \cdots + \frac{|a_3 - a_2|}{|z|^{n-3}} + \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\big\}\Big] \\
\geq{}& |a_n||z|^n\Big[|z| - \frac{1}{|a_n|}\big\{|ra_n - a_{n-1} - ra_n + a_n| \\
&+ |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + |a_{n-3} - a_{n-4}| \\
&+ \cdots + |a_{n-m+1} - a_{n-m}| + |a_{n-m} - a_{n-m-1}| + |a_{n-m-1} - a_{n-m-2}| \\
&+ \cdots + |a_3 - a_2| + |a_2 - a_1| + |a_1 - \delta - a_0 + \delta| + |a_0|\big\}\Big] \\
\geq{}& |a_n||z|^n\Big[|z| - \frac{1}{|a_n|}\big\{(a_{n-1} - ra_n) + (1 - r)|a_n| + (a_{n-1} - a_{n-2}) \\
&+ (a_{n-3} - a_{n-2}) + (a_{n-3} - a_{n-4}) \\
&+ \cdots + (a_{n-m+1} - a_{n-m+2}) + (a_{n-m+1} - a_{n-m}) \\
&+ (a_{n-m-1} - a_{n-m}) + (a_{n-m-2} - a_{n-m-1}) \\
&+ \cdots + (a_2 - a_3) + (a_1 - a_2) + (a_0 + \delta - a_1) + \delta + |a_0|\big\}\Big]
\end{aligned}
$$

if both n and $(n-m)$ are even or odd

$$
\begin{aligned}
&= |a_n||z|^n\Big[|z| - \frac{1}{|a_n|}\big\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta \\
&\quad + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}] \\
&\quad - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}]\big)\big\}\Big] \\
&> 0
\end{aligned}
$$

if

$$|z| > \frac{1}{|a_n|}\Big\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}]\big)\Big\}.$$

This shows that if $|z| > 1$, then $Q(z) > 0$. If

$$|z| > \frac{1}{|a_n|}\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}]\big)\Big\}.$$

Hence all the zeros of $Q(z)$ with $|z| > 1$ lie in

$$|z| \le \frac{1}{|a_n|}\Big\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+3} + a_{n-m+1}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}]\big)\Big\},$$

if both n and $(n - m)$ are even or odd.

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem 1.14, if both n and $(n - m)$ are even or odd.

Similarly we can also prove for n is even and $(n - m)$ is odd (or) if n is odd and $(n - m)$ is even degree polynomials.For this we can rearrange the terms. That is if n is even and $(n - m)$ is odd (or) if n is odd and $(n - m)$ is even then all the zeros $P(z)$ lie in

$$|z| \le \frac{1}{|a_n|}\Big\{|a_n| + |a_0| + a_0 - r(|a_n| + a_n) + 2\delta + 2\big([a_{n-1} + a_{n-3} + \cdots + a_{n-m+4} + a_{n-m+2}] - [a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}]\big)\Big\}.$$

■

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