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Solution of First Order Linear Forward Difference Equation with Variable Coefficients

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Abstract

In this paper, first order linear forward difference equation with variable coefficients be solved with the help of integrating factor.

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1 Introduction:

A difference equation of the form $\Delta y(x) + P(x)y(x) = Q(x) \cdots (1)$, where $P(x)$ and $Q(x)$ are functions of x and contain step size h also, is called first order linear forward difference equation with variable coefficients.

Solution of above equation (1) is given by

$$y(x) = \frac{1}{u(x)}\Delta^{-1}\{u(x+h)Q(x)\} + \frac{c}{u(x)}$$

where c is an arbitrary constant and $u(x)$ is an integrating factor and is given by

$$u(x) = e^{\Delta^{-1}\log\left\{\frac{1}{1-P(x)}\right\}}.$$

Method and Proof:

From above equation (1)

$$\Delta y(x) + P(x)y(x) = Q(x)$$

Multiplying by $u(x+h)$ on both sides, then

$$u(x+h)\Delta y(x) + u(x+h)P(x)y(x) = u(x+h)Q(x)$$

Take $u(x+h)P(x) = \Delta u(x) \cdots (2)$, then

$$u(x+h)\Delta y(x) + y(x)\Delta u(x) = u(x+h)Q(x)$$

We know that

$$\Delta\{\phi(x)\psi(x)\} = \phi(x+h)\Delta\psi(x) + \psi(x)\Delta\phi(x)$$

or

$\phi(x)\Delta\psi(x)+\psi(x+h)\Delta\phi(x)$

Then, $\Delta\{u(x)y(x)\}=u(x+h)Q(x)$

$u(x)y(x)=\Delta^{-1}\{u(x+h)Q(x)\}+c$, where c is an arbitrary constant

$$y(x)=\frac{1}{u(x)}\Delta^{-1}\{u(x+h)Q(x)\}+\frac{c}{u(x)}$$

From (2) $u(x+h)P(x)=\Delta u(x)$

$u(x+h)P(x)=u(x+h)-u(x)$

$u(x)=\{1-P(x)\}u(x+h)$

$\log u(x)=\log\{1-P(x)\}+\log u(x+h)$

$\log u(x+h)-\log u(x)=-\log\{1-P(x)\}$

$$\Delta\log u(x)=\log\left\{\frac{1}{1-P(x)}\right\}$$

$$\log u(x)=\Delta^{-1}\log\left\{\frac{1}{1-P(x)}\right\}$$

$$u(x)=e^{\Delta^{-1}\log\left\{\frac{1}{1-P(x)}\right\}}$$

Thus, $y(x)=\frac{1}{u(x)}\Delta^{-1}\{u(x+h)Q(x)\}+\frac{c}{u(x)}$, where c is an arbitrary constant and $u(x)$ is an integrating factor and is given by

$$u(x)=e^{\Delta^{-1}\log\left\{\frac{1}{1-P(x)}\right\}}.$$

Example 1. Show that $\Delta^{-1}\{f(x)g(x)\}=f(x-h)\Delta^{-1}g(x)-\Delta^{-1}\{\Delta f(x-h)\Delta^{-1}g(x)\}$, where h is step size.

Solution. We know that

$\Delta\{\phi(x)\psi(x)\}=\phi(x)\Delta\psi(x)+\psi(x+h)\Delta\phi(x)$

$\psi(x+h)\Delta\phi(x)=\Delta\{\phi(x)\psi(x)\}-\phi(x)\Delta\psi(x)$

$\Delta^{-1}\{\psi(x+h)\Delta\phi(x)\}=\Delta^{-1}\Delta\{\phi(x)\psi(x)\}-\Delta^{-1}\{\phi(x)\Delta\psi(x)\}$

$\Delta^{-1}\{\psi(x+h)\Delta\phi(x)\}=\phi(x)\psi(x)-\Delta^{-1}\{\phi(x)\Delta\psi(x)\}$

Take $\psi(x+h)=f(x)$ and $\Delta\phi(x)=g(x)$, then

$\Delta^{-1}\{f(x)g(x)\}=f(x-h)\Delta^{-1}g(x)-\Delta^{-1}\{\Delta f(x-h)\Delta^{-1}g(x)\}$.

Example 2. Solve $\Delta y(x)+\frac{h}{x+h}y(x)=x^{(-3)}$, where h is step size.

Solution. $\Delta y(x)+\frac{h}{x+h}y(x)=x^{(-3)}$

Equate with $\Delta y(x)+P(x)y(x)=Q(x)$

Here $P(x)=\dfrac{h}{x+h}$ and $Q(x)=x^{(-3)}$

$$u(x)=e^{\Delta^{-1}\log\left\{\frac{1}{1-P(x)}\right\}}$$
$$=e^{\Delta^{-1}\log\left\{\frac{1}{1-\frac{h}{x+h}}\right\}}$$
$$=e^{\Delta^{-1}\log\left\{\frac{x+h}{x}\right\}}$$
$$=e^{\Delta^{-1}\{\log(x+h)-\log x\}}$$
$$=e^{\Delta^{-1}\Delta\log x}$$
$$=e^{\log x}$$
$$=x$$

Solution is given by

$$y(x)=\frac{1}{u(x)}\Delta^{-1}\{u(x+h)Q(x)\}+\frac{c}{u(x)}\text{, where } c \text{ is an arbitrary constant}$$
$$=\frac{1}{x}\Delta^{-1}\{(x+h)x^{(-3)}\}+\frac{c}{x}$$
$$=\frac{1}{x}\Delta^{-1}\left\{(x+h)\frac{1}{(x+h)(x+2h)(x+3h)}\right\}+\frac{c}{x}$$
$$=\frac{1}{x}\Delta^{-1}\left\{\frac{1}{(x+2h)(x+3h)}\right\}+\frac{c}{x}$$
$$=\frac{1}{x}\Delta^{-1}(x+h)^{(-2)}+\frac{c}{x}$$
$$=\frac{1}{x}\left\{\frac{(x+h)^{(-1)}}{-h}\right\}+\frac{c}{x}$$
$$=-\frac{1}{hx(x+2h)}+\frac{c}{x}$$
$$=\frac{c}{x}-\frac{1}{hx(x+2h)}.$$

Example 3. Solve $\Delta y(x)+\left(1-a^{-h}\dfrac{x-h}{x+h}\right)y(x)=\dfrac{1}{(x+h)^{(2)}}$, where h is step size.

Solution. $\Delta y(x)+\left(1-a^{-h}\dfrac{x-h}{x+h}\right)y(x)=\dfrac{1}{(x+h)^{(2)}}$

Equate with $\Delta y(x)+P(x)y(x)=Q(x)$

Here $P(x)=\left(1-a^{-h}\dfrac{x-h}{x+h}\right)$ and $Q(x)=\dfrac{1}{(x+h)^{(2)}}$

$$u(x)=e^{\Delta^{-1}\log\left\{\frac{1}{1-P(x)}\right\}}$$

$$= e^{\Delta^{-1}\log\left\{\frac{1}{1-\left(1-a^{-h}\frac{x-h}{x+h}\right)}\right\}}$$

$$= e^{\Delta^{-1}\log\left\{\frac{1}{1-1+a^{-h}\frac{x-h}{x+h}}\right\}}$$

$$= e^{\Delta^{-1}\log\left\{\frac{a^h(x+h)}{(x-h)}\right\}}$$

$$= e^{\Delta^{-1}\left\{\log a^h(x+h)-\log(x-h)\right\}}$$

$$= e^{\Delta^{-1}\{h\log a+\log(x+h)-\log(x-h)\}}$$

$$= e^{\Delta^{-1}\{h\log a+\log(x+h)-\log x+\log x-\log(x-h)\}}$$

$$= e^{\Delta^{-1}\{h\log a+\Delta\log x+\Delta\log(x-h)\}}$$

$$= e^{\Delta^{-1}\{h\log a+\Delta\log x(x-h)\}}$$

$$= e^{\Delta^{-1}\{h\log a\}+\Delta^{-1}\Delta\log x^{(2)}}$$

$$= e^{x\log a+\log x^{(2)}}$$

$$= e^{\log a^x+\log x^{(2)}}$$

$$= e^{\log a^x x^{(2)}}$$

$$= a^x x^{(2)}$$

Solution is given by

$$y(x) = \frac{1}{u(x)}\Delta^{-1}\left\{u(x+h)Q(x)\right\} + \frac{c}{u(x)}$$, where c is an arbitrary constant

$$= \frac{1}{a^x x^{(2)}}\Delta^{-1}\left\{a^{x+h}(x+h)^{(2)}\frac{1}{(x+h)^{(2)}}\right\} + \frac{c}{a^x x^{(2)}}$$

$$= \frac{1}{a^x x^{(2)}}\Delta^{-1}\left\{a^{x+h}\right\} + \frac{c}{a^x x^{(2)}}$$

$$= \frac{1}{a^x x^{(2)}}\frac{a^{x+h}}{(a^h-1)} + \frac{c}{a^x x^{(2)}}$$

$$= \frac{1}{x^{(2)}}\frac{a^h}{(a^h-1)} + \frac{c}{a^x x^{(2)}}.$$

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