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Fixed Point Results with a New Type of G-iteration in Normed Linear Spaces

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Abstract

In this paper a new type of one step iteration for self mappings is introduced under certain conditions in normed linear space and studied with quasi contractive mapping and quasi contractive pair of mappings.

Keywords:-Common fixed point, Contractive condition, G-iteration, Mann iteration.

Mathematics Subject Classification:- 47H10,54H25.

1.INTRODUCTION

Let X be a nonempty closed convex subset of a normed linear space E and $T : X \rightarrow X$ be a self mapping and $\{x_n\}$ be the sequence then for arbitrary $x_0 \in X$ Mann[4] iteration process is defined as

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n \quad for\ n \geq 0$$

Similarly Ishikawa[3] iteration process for $\{x_n\}$ is given by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T y_n$$

and $y_n = (1-\lambda_n^{'})x_n + \lambda_n^{'} T x_n$, for $n \geq 0$

Where $x_0 \in X$ is arbitrary and $\{\lambda_n\},\{\lambda_n^{'}\}$ are sequences of real numbers such that

$0 \le \lambda_n \le 1$, $0 \le \lambda_n^{'} \le 1$.

By using the concept of Mann iteration process Sahu[5] introduced a new G-iteration process as follows:

Let T be a self mapping of Banach space then G-iteration process associated by T is defined as, Let $x_0, x_1 \in X$ and

$$x_{n+2} = (\mu_n - \lambda_n) x_{n+1} + \lambda_n T x_{n+1} + (1 - \mu_n) T x_n \quad \text{for } n \ge 0$$

Where $\{\mu_n\}$ and $\{\lambda_n\}$ satisfy

(i) $\lambda_0 = \mu_0 = 1$

(ii) $0 < \lambda_n < 1, n > 0$ and $\mu_n \ge \lambda_n$ for $n \ge 0$

(iii) $\lim_{n\to\infty} \lambda_n = h > 0$

(iv) $\lim_{n\to\infty} \mu_n = 1$

Das and Debata [1] generalized the Ishikawa iteration processes from the case of one self mapping to the case of two self mappings S and T of X given by

$$x_{n+1} = (1 - \lambda_n) x_n + \lambda_n S y_n$$

$$\text{and } y_n = (1 - \lambda_n^{'}) x_n + \lambda_n^{'} T x_n , \quad \text{for } n \ge 0$$

By using above iteration Das and Debata[1] established the common fixed points of quasi-non expansive mappings in a uniformly convex Banach space.Several other researchers such as Takahashi and Tamura[6] investigated iteration in a strictly convex Banach space,for the case of two nonexpansive mappings under different assumptions and contractive conditions.

In this paper a new type of one step iteration for self mappings is introduced and studied with a contractive type conditions of Sahu[5].The result obtained in this paper extend and improve the corresponding results of Dhage[2] and Sahu [5].

2. PRELIMINARIES

Theorem 2.1: Dhage[2] has proved a fixed point theorem satisfying the inequality

$$\|Tx - Ty\| \le a(\|x - Tx\| + \|y - Ty\|)$$
$$+ (1-2a)\max\left\{\|x-y\|, \|x-Ty\|, \|y-Ty\|, \frac{1}{2}(\|x-Tx\|+\|y-Ty\|), \frac{1}{2}(\|x-Ty\|+\|y-Tx\|)\right\}$$

Definition 2.3: Let X be a normed space and $T : X \to X$ is a self mapping then T is said to satisfy a Lipschitz condition with constant q if

$$\|Tx - Ty\| \le q\|x - y\| \qquad \forall x, y \in X$$

If $q < 1$ then T is called a contraction mapping.

Definition 2.4: Let X be a normed space then a self mapping T of X is called quasi contractive mapping if

$$\|Tx - Ty\| \le q \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}$$

$$\forall x, y \in X, \ where\ 0 < q < 1.$$

Sahu[5]extend the above definition for a pair of mapping in the following manner:-

Definition 2.5: Let X be a normed space then T_1 and T_2 be two self mappings of X is called quasi contractive pair of mapping if

$$\|T_1x - T_2y\| \le q \max\{\|x - y\|, \|x - T_1x\|, \|y - T_2y\|, \|x - T_2y\|, \|y - T_1x\|\}$$

$$\forall x, y \in X, \ where\ 0 < q < 1.$$

Recursion-2.6

Let $x_0, x_1 \in X$ and

$$x_{n+2} = (\mu_n - \lambda_n - s_n)x_{n+1} + (\lambda_n + s_n)Tx_{n+1} + (1 - \mu_n - \lambda_n + k_n)Tx_n + (\lambda_n - k_n)x_n \text{ for } n \ge 0$$

where $\{\mu_n\}, \{\lambda_n\}, \{k_n\}$ and $\{s_n\}$ satisfying

(i) $\mu_0 = \lambda_0 = k_0 = 1$

(ii) $0 < \lambda_n < 1$, $0 < k_n < 1$, $0 < s_n < 1$ for $n > 0$.

(iii) $\mu_n \ge \lambda_n, \mu_n \ge k_n$, $\mu_n \ge s_n$ for $n \ge 0$.

(iv) $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} k_n = \xi$ where $\xi > 0$.

(v) $\lim_{n\to\infty} \mu_n = 1$.

Recursion-2.7

Let $x_0, x_1 \in X$ and

$$x_{2n+2} = (\mu_n - \lambda_n - s_n)x_{2n+1} + (\lambda_n + s_n)T_1x_{2n+1} + (1-\mu_n - \lambda_n + k_n)T_2x_{2n} + (\lambda_n - k_n)x_{2n}$$

and $x_{2n+3} = (\mu_n - \lambda_n - s_n)x_{2n+2} + (\lambda_n + s_n)T_2x_{2n+2} + (1-\mu_n - \lambda_n + k_n)T_1x_{2n+1} + (\lambda_n - k_n)x_{2n+1}$, for $n \geq 0$

where $\{\mu_n\}, \{\lambda_n\}, \{k_n\}$ and $\{s_n\}$ satisfying

(i) $\mu_0 = \lambda_0 = k_0 = 1$

(ii) $0 < \lambda_n < 1$, $0 < k_n < 1, 0 < s_n < 1$ for $n > 0$.

(iii) $\mu_n \geq \lambda_n, \mu_n \geq k_n, \mu_n \geq s_n$ for $n \geq 0$.

(iv) $\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} k_n = \xi$ where $\xi > 0$.

(v) $\lim_{n\to\infty} \mu_n = 1$.

3.MAIN RESULTS

Theorem 3.1: Let X be a closed convex subset of normed linear space N and let $T: X \to X$ be a quasi contractive self mapping of X. Suppose that $\{\mu_n\}, \{\lambda_n\}, \{k_n\}$ *and* $\{s_n\}$ are real sequences in $[0,1]$. For two arbitrary $x_0, x_1 \in X$ define the sequence $\{x_n\}$ by the recursion (2.6). If $\lim_{n\to\infty} x_n = z \in X$ then z is the fixed point of T.

Proof:- If $\{x_n\}$ converges on $z \in X$ i.e. $\lim_{n\to\infty} x_n = z$.

We shall show that z is the fixed point of T .

Consider

$$\begin{aligned}
\|z - Tz\| &\leq \|z - x_{n+2}\| + \|x_{n+2} - Tz\| \\
&\leq \|z - x_{n+2}\| + \|(\mu_n - \lambda_n - s_n)x_{n+1} + (\lambda_n + s_n)Tx_{n+1} + (1-\mu_n - \lambda_n + k_n)Tx_n + (\lambda_n - k_n)x_n - Tz\| \\
&\leq \|z - x_{n+2}\| + (\mu_n - \lambda_n - s_n)\|x_{n+1} - Tz\| + (\lambda_n + s_n)\|Tx_{n+1} - Tz\| + (1-\mu_n - \lambda_n + k_n)\|Tx_n - Tz\| \\
&\qquad + (\lambda_n - k_n)\|x_n - Tz\| \\
&\leq \|z - x_{n+2}\| + (\mu_n - \lambda_n - s_n)\|x_{n+1} - Tz\| \\
&\qquad + (\lambda_n + s_n) q \max\{\|x_{n+1} - z\|, \|x_{n+1} - Tx_{n+1}\|, \|z - Tz\|, \|x_{n+1} - Tz\|, \|z - Tx_{n+1}\|\} \ldots \quad (3.1.1) \\
&\qquad + (1-\mu_n - \lambda_n + k_n)\|Tx_n - Tz\| + (\lambda_n - k_n)\|x_n - Tz\|
\end{aligned}$$

We observe by the definition of G-iteration that

$$\|x_{n+1}-Tx_{n+1}\|\leq\frac{1}{(\lambda_n+s_n)}\|x_{n+1}-x_{n+2}\|+\frac{(1-\mu_n)}{(\lambda_n+s_n)}\|x_{n+1}-Tx_n\|+\frac{(\lambda_n-k_n)}{(\lambda_n+s_n)}\|x_n-Tx_n\|$$

and

$$\|z-Tx_{n+1}\|\leq\|z-x_{n+1}\|+\|x_{n+1}-Tx_{n+1}\|$$

$$\leq\|z-x_{n+1}\|+\frac{1}{(\lambda_n+s_n)}\|x_{n+1}-x_{n+2}\|+\frac{(1-\mu_n)}{(\lambda_n+s_n)}\|x_{n+1}-Tx_n\|+\frac{(\lambda_n-k_n)}{(\lambda_n+s_n)}\|x_n-Tx_n\|$$

Now putting above values in (3.1.1) then we have

$$\|z-Tz\|\leq\|z-x_{n+2}\|+(\mu_n-\lambda_n-s_n)\|x_{n+1}-Tz\|$$

$$+(\lambda_n+s_n)q\max\begin{Bmatrix}\|x_{n+1}-z\|,\frac{1}{(\lambda_n+s_n)}\|x_{n+1}-x_{n+2}\|+\frac{(1-\mu_n)}{(\lambda_n+s_n)}\|x_{n+1}-Tx_n\|+\frac{(\lambda_n-k_n)}{(\lambda_n+s_n)}\|x_n-Tx_n\|,\|z-Tz\|,\\ \|x_{n+1}-Tz\|,\|z-x_{n+1}\|+\frac{1}{(\lambda_n+s_n)}\|x_{n+1}-x_{n+2}\|+\frac{(1-\mu_n)}{(\lambda_n+s_n)}\|x_{n+1}-Tx_n\|+\frac{(\lambda_n-k_n)}{(\lambda_n+s_n)}\|x_n-Tx_n\|\end{Bmatrix}$$

$$+(1-\mu_n-\lambda_n+k_n)\|Tx_n-Tz\|+(\lambda_n-k_n)\|x_n-Tz\|$$

Letting $n\to\infty$ then we have

$$\|z-Tz\|\leq(1-2\xi+2\xi q)\|z-Tz\|$$

$$\Rightarrow\|z-Tz\|=0 \text{ Since } 0<q<1 \text{ and } \xi>0$$

Hence $z=Tz$ is a fixed point of T.

Remark: When $\{S_n\}=\{0\},\{\mu_n\}=\{1\}$ and $\{\lambda_n\}=\{k_n\}$ then above G-iterative process reduces to Mann iteration.

Theorem 3.2: Let X be a closed convex subset of normed linear space N and let T_1 and T_2 be quasi contractive pair of self mappings of X. Suppose that $\{\mu_n\},\{\lambda_n\},\{k_n\}$ *and* $\{s_n\}$ are real sequences in $[0,1]$. For two arbitrary $x_0,x_1\in X$ define the sequence $\{x_n\}$ by the recursion (2.7). If $\lim_{n\to\infty}x_n=z\in X$ then z is the common fixed point of T_1 and T_2.

Proof:-If $\{x_n\}$ converges on $z\in X$

i.e. $\lim_{n\to\infty} x_n = z$.

We shall show that z is the fixed point of T .

Consider

$$\begin{aligned}
\|z-T_1z\| &\le \|z-x_{2n+3}\| + \|x_{2n+3}-T_1z\| \\
&\le \|z-x_{2n+3}\| + \|(\mu_n-\lambda_n-s_n)x_{2n+2} + (\lambda_n+s_n)T_2x_{2n+2} + (1-\mu_n-\lambda_n+k_n)T_1x_{2n+1} + (\lambda_n-k_n)x_{2n+1} - T_1z\| \\
&\le \|z-x_{2n+3}\| + (\mu_n-\lambda_n-s_n)\|x_{2n+2}-T_1z\| + (\lambda_n+s_n)\|T_2x_{2n+2}-T_1z\| + (1-\mu_n-\lambda_n+k_n)\|T_1x_{2n+1}-T_1z\| \\
&\qquad + (\lambda_n-k_n)\|x_{2n+1}-T_1z\| \\
&\le \|z-x_{2n+3}\| + (\mu_n-\lambda_n-s_n)\|x_{2n+2}-T_1z\| \\
&\quad + (\lambda_n+s_n)q\max\{\|x_{2n+2}-z\|, \|x_{2n+2}-T_2x_{2n+2}\|, \|z-T_1z\|, \|z-T_2x_{2n+2}\|, \|x_{2n+2}-T_1z\|\} \quad (3.2.1) \\
&\quad + (1-\mu_n-\lambda_n+k_n)\|T_1x_{2n+1}-T_1z\| + (\lambda_n-k_n)\|x_{2n+1}-T_1z\|
\end{aligned}$$

We observe by the definition of G-iteration that

$$\|x_{2n+2}-T_2x_{2n+2}\| \le \frac{1}{(\lambda_n+s_n)}\|x_{2n+2}-x_{2n+3}\| + \frac{(1-\mu_n)}{(\lambda_n+s_n)}\|x_{2n+2}-T_1x_{2n+1}\| + \frac{(\lambda_n-k_n)}{(\lambda_n+s_n)}\|x_{2n+1}-T_1x_{2n+1}\|$$

and

$$\begin{aligned}
\|z-T_2x_{2n+2}\| &\le \|z-x_{2n+2}\| + \|x_{2n+2}-T_2x_{2n+2}\| \\
&\le \|z-x_{2n+2}\| + \frac{1}{(\lambda_n+s_n)}\|x_{2n+2}-x_{2n+3}\| + \frac{(1-\mu_n)}{(\lambda_n+s_n)}\|x_{2n+2}-T_1x_{2n+1}\| + \frac{(\lambda_n-k_n)}{(\lambda_n+s_n)}\|x_{2n+1}-T_1x_{2n+1}\|
\end{aligned}$$

Now putting above values in (3.2.1) then we have

$$\begin{aligned}
\|z-T_1z\| &\le \|z-x_{2n+3}\| + (\mu_n-\lambda_n-s_n)\|x_{2n+2}-T_1z\| \\
&\quad + (\lambda_n+s_n)q\max\left\{\begin{array}{l}
\|x_{2n+2}-z\|, \|z-T_1z\|, \\
\frac{1}{(\lambda_n+s_n)}\|x_{2n+2}-x_{2n+3}\| + \frac{(1-\mu_n)}{(\lambda_n+s_n)}\|x_{2n+2}-T_1x_{2n+1}\| + \frac{(\lambda_n-k_n)}{(\lambda_n+s_n)}\|x_{2n+1}-T_1x_{2n+1}\|, \\
\|z-x_{2n+2}\| + \frac{1}{(\lambda_n+s_n)}\|x_{2n+2}-x_{2n+3}\| + \frac{(1-\mu_n)}{(\lambda_n+s_n)}\|T_1x_{2n+1}-x_{2n+2}\| + \frac{(\lambda_n-k_n)}{(\lambda_n+s_n)}\|x_{2n+1}-T_1x_{2n+1}\|, \\
\|x_{2n+2}-T_1z\|
\end{array}\right\} \\
&\quad + (1-\mu_n-\lambda_n+k_n)\|T_1x_{2n+1}-T_1z\| + (\lambda_n-k_n)\|x_{2n+1}-T_1z\|
\end{aligned}$$

Letting $n\to\infty$ then we have

$$\|z-T_1z\| \le (1-2\xi+2\xi q)\|z-T_1z\|$$

$\Rightarrow \|z - T_1 z\| = 0$ Since $0 < q < 1$ and $\xi > 0$

Hence $z = T_1 z$ is a fixed point of T_1.
Similarly we can show that
$\|z - T_2 z\| \le (1 - 2\xi + 2\xi q)\|z - T_2 z\|$
$\Rightarrow \|z - T_2 z\| = 0$ Since $0 < q < 1$ and $\xi > 0$

Hence $z = T_2 z$ z is a fixed point of T_2.
Finally we can say that z is a common fixed point of T_1 & T_2.
This completes the proof of theorem.

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