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A fixed point theorem for four self maps on a Menger space satisfying a convex inequality

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Abstract

In this paper we prove a common fixed point theorem for occasionally weakly compatible self maps in Menger Spaces, satisfying a convex inequality which was not considered before. We also show that the results of B.D.Pant and S.Chauhan [1] are not valid.

Keywords. Menger Space, Occasionally Weakly Compatible, Heaviside function, convex inequality

1. Introduction

The notion of probabilistic metric space (PM-space), as a generalization of metric spaces, with non deterministic distance, was introduced by K.Menger [6] in 1942. Since then the theory of PM-spaces has developed in many directions [2, 3]. In 1972, V.M.Sehgal and A.T.Bharucha Reid [14] initiated the study of contraction mappings in PM-spaces which is an important step in the development of fixed point theorems.

Various mathematicians weakened the notion of commutativity by introducing the notions of weak commutativity [11], compatibility [5] and weak compatibility [6] in metric spaces and proved a number of fixed point theorems using these notions. Al-Thagafi and Shazad [10] weakened the notion of weakly compatible maps by introducing occasionally weakly compatible maps. It is worth to mention that every pair of commuting self-maps is weakly compatible and each pair of weakly compatible self-maps is occasionally weakly compatible but the reverse is not always true. Many authors formulated the definitions of weakly commuting [12], compatible [13], weakly compatible maps [4] and occasionally weakly compatible maps [8] in probabilistic settings and proved a number of fixed point theorems in this direction.

B.D.Pant and S.Chauhan [1] claimed common fixed point theorems for occasionally weakly compatible self mappings in Menger Spaces. We show that the results are not valid, through examples. We also introduce a convex inequality for four self maps on a Menger Space and establish a common fixed point theorem for such maps. We also obtain a corollary to this result.

An open problem is also given at the end.

2. Preliminaries

Definition 2.1: (B.Schweizer and A.Sklar [3]) A triangular norm * (t-norm) is a binary operation on the unit interval $[0,1]$ such that for all $a, b, c, d \in [0,1]$ the following conditions are satisfied:

1. $a * 1 = a$
2. $a * b = b * a$
3. $a * b \leq c * d$; $whenever\ a \leq c\ and\ b \leq d$
4. $a * (b * c) = (a * b) * c$

Example: Define $a * b = min\{a, b\}$. Then * is a t-norm and $*: [0,1] \times [0,1] \to [0,1]$ is also continuous. Thus * = min is a continuous t-norm.

Definition 2.2: (B.Schweizer and A.Sklar [3]) A mapping $F: R \to [0,1]$ is said to be a distribution function if it is non-decreasing and left continuous with

$$inf\{F(t) : t \in R\} = 0 \text{ and } Sup\{F(t): t \in R\} = 1.$$

We shall denote by $\mathfrak{F}$ the set of all distribution functions.

Define H (t) $= \begin{cases} 0 & if\ t \leq 0 \\ 1 & if\ t > 0 \end{cases}$

Then H is clearly a distribution function which is also called the Heaviside function.

If X is a non empty set, $F: X \times X \to \mathfrak{F}$ is called a Probabilistic distance on X and the value of F at $(x, y) \epsilon X \times X$ is represented by $F_{x,y}$

Definition 2.3: (B.Schweizer and A.Sklar [3]) The ordered pair (X, F) is called a Probabilistic metric space (PM-space) if X is a non empty set and F is a probabilistic distance satisfying the following conditions for all $x, y, z \epsilon X$; $t, s > 0$

1. $F_{x,y}$ = H iff $x = y$
2. $F_{x,y} = F_{y,x}$

3. If $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t + s) = 1$

The ordered triple $(X, F, *)$ is called a Menger Space if (X, F) is a PM-space, * is a t-norm and the following inequality holds

$F_{x+z}(t + s) \geq F_{x,y}(t) * F_{y,z}(s)$ for all $x \epsilon X;\ t, s > 0$

Definition 2.4: (B.Schweizer and A.Sklar [3]) Let $(X, F, *)$ be a Menger space with continuous t-norm *

1. A sequence $\{x_n\}$ in X is said to converge to a point x in X if for every $\varepsilon > 0$, $\lambda > 0$, there exists a positive integer $N(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1-\lambda$ for $all\ n \geq N(\varepsilon, \lambda)$
2. A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$, $\lambda > 0$,there exists a positive integer $N(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1-\lambda$ for all n, m $\geq N(\varepsilon, \lambda)$
3. A Menger space in which every Cauchy sequence is convergent, is said to be complete.

Definition 2.5: (S.N.Mishra [13]) Self maps A and B of a Menger space $(X, F, *)$ are said to be compatible if $F_{ABx_n,BAx_n}(t) \to 1$ for all t > 0, whenever $\{x_n\}$is a sequence in X such that Ax_n, $Bx_n \to$ z for some z in X as $n \to \infty$

Lemma 2.1 (S.N.Mishra [13]) Let $(X, F, *)$ be a Menger space with continuous t-norm * If there exists a constant $k \epsilon$ (0, 1) such that $Fx, y(kt) \geq Fx, y(t)$ for all $x, y \in$ X and t > 0 then $x = y$

Definition 2.6: (B.Singh and S.Jain [4]) Self maps A and B of a non empty set X are said to be weakly compatible (coincidentally commuting) if they commute at their coincidence points if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$. In this case, $w = Ax = Bx$ is called a point of coincidence of A and B.

Definition 2.7: (G.Jungck and B.E.Rhoades [7]) Self maps A and B of a none empty set X are occasionally weakly compatible if and only if there is a point $x \in X$ which is a coincidence point of A and B at which A and B commute.

Lemma 2.2 (G.Jungck and B.E.Rhoades [7]) Let X be a non empty set and A and B be occasionally weakly compatible self maps of X. If A and B have a unique point of coincidence, $w = Ax = Bx$, then w is the unique common fixed point of A and B.

B.D.Pant and S.Chauhan [1] claimed the following theorems

Theorem 2.1 ([1], Theorem 3.1) Let $(X, F, *)$ be a Menger space with * = min. Suppose A, B, S and T are self maps on X. Further, let the pairs (A, S) and (B, T) be occasionally weakly compatible in X satisfying

$\left[1 + aF_{Sx,Ty}(kt)\right] * F_{Ax,By}(kt) \geq$

$$a\,min\begin{pmatrix} F_{Ax,Sx}(kt) * F_{By,Ty}(kt) \\ F_{Ax,Ty}(2kt) * F_{By,Sx}(2kt) \end{pmatrix} + \begin{pmatrix} F_{Sx,Ty}(t) * F_{Ax,Sx}(t) * F_{By,Ty}(t) * \\ F_{Ax,Ty}(2t) * F_{By,Sx}(2t) \end{pmatrix} \to (2.1.1)$$

for all t > 0 and x,y∈X with fixed constants $a \in (-1,0]$ and $k \in (0,1)$.

Then there exists a unique point w∈X such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover $z = w$ so that there is a unique common fixed point of A, B, S and T

Theorem 2.2 ([1], Theorem 3.2) Let $(X, F, *)$ be a Menger Sapce with continuous t-norm *=min. Further let the pair (A, S) is occasionally weakly compatible in X satisfying

$$[1 + aF_{Sx,Sy}(kt)] * F_{Ax,Ay}(kt) \geq$$
$$a\,min\begin{pmatrix} F_{Ax,Sx}(kt) * F_{Ay,Sy}(kt) \\ F_{Ax,Sy}(2kt) * F_{Ay,Sx}(2kt) \end{pmatrix} + \begin{pmatrix} F_{Sx,Sy}(t) * F_{Ax,Sx}(t) * F_{Ay,Sy}(t) * \\ F_{Ax,Sy}(2t) * F_{Ay,Sx}(2t) \end{pmatrix} \to (2.2.1)$$

for all t > 0, and $x, y \in X$ with fixed constants $a \in (-1,0]$ and $k \in (0,1)$. Then A and S have a unique common fixed point in X.

In the next section, we show that both the theorems are not valid, by providing examples.

3. Examples and Main Results

In this section we first give two examples which show that Theorem 2.1 and Theorem 2.2 are not valid.

Example 3.1 Let $X = \{0,1\}$; $A = S = I$ (Identity map); $B(0) = 1, B(1) = 0, T = B$, $Fx, y(t) = H(t - |x - y|)$ for all $x, y \in X$. Then clearly, all the hypotheses of Theorem 2.1 except (2.1.1) are satisfied. Now we show that (2.1.1) is also satisfied with $k \in (0,1)$ and $a \in (-1,0)$.

Inequality (2.1.1) becomes

$$[1 + aF_{x,By}(kt)] * F_{x,By}(kt) \geq$$
$$a\,min\begin{pmatrix} F_{x,x}(kt) * F_{By,By}(kt) \\ F_{x,By}(2kt) * F_{By,x}(2kt) \end{pmatrix} + \begin{pmatrix} F_{x,By}(t) * F_{x,x}(t) * F_{By,By}(t) * \\ F_{x,By}(2t) * F_{By,x}(2t) \end{pmatrix}$$

$$\Rightarrow [1 + aH(kt - |x - By|)] * H(kt - |x - By|) \geq$$
$$aH(2kt - |x - By|) + H(t - |x - By|) \to (3.1.1)$$

Suppose $x = 0$ and $y = 1$. Then

$$[1 + aH(kt)] * H(kt) \geq aH(2kt) + H(t)$$
$$\Rightarrow (1 + a) * 1 \geq a + 1 \qquad (since\ H(kt) = 1\ if\ t > 0)$$

This shows that (3.1.1) holds.

Suppose $y = x$. Then $By = Bx$ and (3.1.1) becomes

$$[1 + aH(kt - |x - Bx|)] * H(kt - |x - Bx|) \geq$$
$$aH(2kt - |x - Bx|) + H(t - |x - Bx|)$$
$$\Rightarrow [1 + aH(kt - 1)] * H(kt - 1) \geq aH(2kt - 1) + H(t - 1) \to (3.1.2)$$

Case (i): $\boldsymbol{H\ (kt-1)\ =\ 0}$.
Then L.H.S of (3.1.2) is 1 while R.H.S of (3.1.2) is $0\ or\ a+1\ or\ 1\ or\ a$.
Since $-1\ <\ a\ \leq\ 0$, this shows that L.H.S ≥ R.H.S. Thus (3.1.2) holds

Case (ii): $\boldsymbol{H\ (kt-1)\ =\ 1}$. **Then** $\boldsymbol{kt-1\ >\ 0}$
$\Rightarrow\ kt\ >\ 1\ \Rightarrow\ 2kt\ >\ kt\ >\ 1\ and\ t\ >\ kt\ >\ 1\ (since\ 0\ <\ k\ <1)$
$\therefore\ H\ (2kt-1)\ =\ 1$ and $H\ (t-1)\ =\ 1$.

This shows that clearly (3.1.2) holds.

Thus (3.1.1) holds always. But A and S have two common fixed points 0 and 1, while B and T do not have common fixed points.
$\therefore\ A, B, S$ and T have no common fixed points. Thus theorem 2.1 fails.

The following example shows that theorem 2.2 fails

Example 3.2 Let $X\ =\ \{0,1\}$; $A\ =\ S\ =\ I\ (Identity)$, $*$ = minimum t-norm and $F_{x,y}(\mathrm{t})\ =\dfrac{t}{t+|x-y|}$, $\quad t>0$. Now we show that (2.2.1) holds if $a\ =\ -1/2$ and $k\ =\ ½$

Clearly (2.2.1) becomes

$$[1+aF_{x,y}(kt)]*F_{x,y}(kt)\geq$$
$$a\min\begin{pmatrix} F_{x,x}(kt)*F_{y,y}(kt) \\ F_{x,y}(2kt)*F_{y,x}(2kt)\end{pmatrix}+\begin{pmatrix} F_{x,y}(t)*F_{x,x}(t)*F_{y,y}(t)* \\ F_{x,y}(2t)*F_{y,x}(2t)\end{pmatrix}$$
$$=aF_{x,y}(2kt)+\ F_{x,y}(t)\rightarrow(3.2.1)$$

If $x\ =\ y$ the above inequality clearly holds.

Suppose $x\ \neq\ y$. We may suppose that $x\ =\ 0, y\ =\ 1$.

Then $F_{x,y}(\mathrm{t})=\dfrac{t}{t+|x-y|}\ =\ \dfrac{t}{t+1},\qquad t>0$

Then (3.2.1) becomes

$$\left[1+a\frac{kt}{kt+1}\right]*\frac{kt}{kt+1}\geq a\frac{2kt}{2kt+1}+\frac{t}{t+1}$$

Put $k\ =\ ½$

$$\left[1+a\frac{t/2}{\frac{t}{2}+1}\right]*\frac{t/2}{\frac{t}{2}+1}\geq a\frac{t}{t+1}+\frac{t}{t+1}=(a+1)\frac{t}{t+1}$$
$$\Rightarrow\left[1+\frac{ta}{t+2}\right]*\frac{t}{t+2}\geq(a+1)\frac{t}{t+1}$$

Observe that $1+\frac{ta}{t+2}\geq(a+1)\frac{t}{t+1}$ when $a\ =\ -1/2$ and t > 0.

Also observe that $\frac{t}{t+2}\geq(a+1)\frac{t}{t+1}$ holds when $a\ =\ -1/2$ and t >0.

Hence $\left[1+\frac{ta}{t+2}\right]*\frac{t}{t+2}\geq(a+1)\frac{t}{t+1}$ holds when $a\ =\ -1/2$ and $k\ =\ ½$.

$\therefore$ (3.2.1) holds when k = ½ and $a = -1/2$ and $F_{x,y}(t) = \dfrac{t}{t+|x-y|}$ with * as minimum t-norm. But A and B do not have unique common fixed point. Thus Theorem 2.2 fails.

Now we give a more general example to show the non- validity of Theorem 2.2

Example 3.3 Let X be a subset of R with more than one element. Let S be any self map on X and $A = S$. Take $k = ½$ and $a = -½$ and $F_{x,y}(t) = \dfrac{t}{t+|x-y|}$ with * as a minimum t-norm. Then $(X, F, *)$ is a Menger space, all the hypotheses of Theorem 2.2 is satisfied. But the conclusion of theorem 2.2 does not hold. In fact, every point of X is a coincidence point of A and S. But, depending on the choice of the function A,

1. (A, S) may not have common fixed point
2. (A, S) may have more than one common fixed point.

Now we prove the following Theorem in which the self maps are assumed to satisfy a convex type inequality. It may be mentioned that this type of inequalities are not considered before.

Theorem 3.4 Let $(X, F, *)$ be a Menger space with $*$ = min t-norm, so that * is a continuous t-norm. Suppose $0 < k < 1$; $0 \leq \alpha, \beta \leq 1, \alpha + \beta = 1$. Suppose A, B, S and T be self maps on X. Further let the pairs (A, S) and (B, T) be occasionally weakly compatible in X satisfying

$$[\alpha F_{Sx,Ty}(kt)] * [\beta F_{Ax,By}(kt)] \geq$$
$$\alpha \begin{pmatrix} F_{Ax,Sx}(kt) * F_{By,Ty}(kt) * \\ F_{Ax,Ty}(2kt) * F_{By,Sx}(2kt) \end{pmatrix} + \beta \begin{pmatrix} F_{Sx,Ty}(t) * F_{Ax,Sx}(t) * F_{By,Ty}(t) * \\ F_{Ax,Ty}(2t) * F_{By,Sx}(2t) \end{pmatrix} \rightarrow (3.4.1)$$

for all t > 0 and $x, y \in X$.

Then there exists unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. More over $z = w$ so that there is a unique common fixed point of A, B, S and T.

Proof: Since the pairs (A, S) and (B, T) are occasionally weakly compatible, there exist $x, y \in X$ such that $Ax = Sx$ and $By = Ty$ Inequality (3.4.1) becomes

$$[\alpha F_{Ax,By}(kt)] * [\beta F_{Ax,By}(kt)] \geq$$
$$\alpha \begin{pmatrix} F_{Ax,Ax}(kt) * F_{By,By}(kt) * \\ F_{Ax,By}(2kt) * F_{By,Ax}(2kt) \end{pmatrix} + \beta \begin{pmatrix} F_{Ax,By}(t) * F_{Ax,Ax}(t) * F_{By,By}(t) * \\ F_{Ax,By}(2t) * F_{By,Ax}(2t) \end{pmatrix}$$
$$\Rightarrow (\alpha + \beta) F_{Ax,By}(kt) \geq \alpha F_{Ax,By}(2kt) + \beta F_{Ax,By}(t)$$
$$\geq \begin{cases} \alpha F_{Ax,By}(2kt) + \beta F_{Ax,By}(2kt) & if\ 2k \leq 1 \\ \alpha F_{Ax,By}(t) + \beta F_{Ax,By}(t) & if\ 2k > 1 \end{cases}$$

$$\therefore \quad F_{Ax,By}(kt) \ \geq \ \begin{cases} F_{Ax,By}(2kt) & if\ 2k \leq 1 \\ F_{Ax,By}(t) & if\ 2k > 1 \end{cases}$$

This being true for all $t > 0, Ax = By$ (by Lemma 2.1)
$\therefore Ax = Sx = By = Ty = w$ (say) → (3.4.2)

Since (A, S) is occasionally weakly compatible at x and $Ax = Sx$,
we have $ASx = SAx$ so that $Aw = S$. Hence by the above and by (3.4.2) we get

$$Aw = Sw = By = Ty = w \rightarrow (3.4.3)$$

$\therefore Aw = Sw = w, w$ is a fixed point of $A\ and\ S$.

Since (B, T) is occasionally weakly compatible at y and $By = Ty$,
we have $BTy = TBy$ so that $Bw = Tw$, hence by the above, we get

$$Aw = Sw = Bw = Tw = w \qquad \text{(by (3.4.3))}$$

$\therefore Bw = Tw = w, w$ is a fixed point of B and T.

Uniqueness: Let z be a common fixed point of $A, B, S\ and\ T$. Then put $x = z, y = w$ in (3.4.1) we have

$$[\alpha F_{Az,Bw}(kt)] * [\beta F_{Az,Bw}(kt)] \geq$$
$$\alpha \begin{pmatrix} F_{Az,Az}(kt) * F_{Bw,Bw}(kt) * \\ F_{Az,Bw}(2kt) * F_{Bw,Az}(2kt) \end{pmatrix} + \beta \begin{pmatrix} F_{Az,Bw}(t) * F_{Az,Az}(t) * F_{Bw,Bw}(t) * \\ F_{Az,Bw}(2t) * F_{Bw,Az}(2t) \end{pmatrix}$$
$$\Rightarrow (\alpha + \beta) F_{z,w}(kt) \geq \alpha F_{z,w}(2kt) + \beta F_{z,w}(t)$$
$$\therefore \quad F_{Ax,By}(kt) \ \geq \ \begin{cases} F_{z,w}(2kt) & if\ 2k \leq 1 \\ F_{z,w}(t) & if\ 2k > 1 \end{cases}$$

This being true for all t > 0, $z = w$
$\therefore w$ is the unique common fixed point for $A, B, S\ and\ T$.

From Theorem 3.4 with $\alpha = 0,\ \beta = 1$ we have the following Corollary.

Corollary 3.5 Let $(X, F, *)$ be a Menger space with continuous t-norm, *=min. Further let $(A, S)\ and\ (B, T)$ be occasionally weakly compatible in X satisfying

$$F_{Ax,By}(kt) \geq \begin{pmatrix} F_{Sx,Ty}(t) * F_{Ax,Sx}(t) * F_{By,Ty}(t) * \\ F_{Ax,ty}(2t) * F_{By,Sx}(2t) \end{pmatrix}$$

for all $t > 0$; $x, y \in X$ with a constant $k \in (0, 1)$. Then there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. More over $z = w$ so that there is a unique common fixed point of A, B, S and T.

Open Problem: Is Theorem 3.4 true if min. t-norm is replaced by any continuous t-norm?

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