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On the Binding Number of Middle Graph of Graphs

H.B. Walikar

Presently serving as Vice Chancellor,
Karnatak University, Dharwad, Karnataka

B.B. Mulla

Department of Mathematics,
Smt. Indira Gandhi College of Engineering,
Sector-16, Koparkhairane, Navi Mumbai 400709
E-mail: bbm6@yahoo.com

Abstract

The binding number of a graph was introduced by D.R.Woodall in 1973 [10] and is defined as the minimum of the ratios $|\Gamma(X)|/|X|$ taken over all non-empty subsets of X of vertices in G such that $\Gamma(X) \neq V(G)$, where $\Gamma(X) = \cup_{v\in X}\Gamma(v)$ and $\Gamma(v)$ the set of all vertices adjacent to a vertex v in G. We obtain exact values of the binding number of middle graph of cycle, path, complete graph and complete bipartite graph.

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1. Introduction

We consider only finite simple graphs G with vertex set $V(G)$ and edge set $E(G)$. For a graph $G = (V, E)$ and a set $X \subseteq V$, we denote by $\Gamma(X)$ the set of vertices joined to vertices in X. A set of independent edges which cover all vertices of a graph is called 1-factor of a graph. By (1,2)-factor of a graph G, we mean a set of independent edges or vertex disjoint cycles which cover all vertices of G. Clearly, the cycles in the definition are of odd length. A graph G is hallian, if $|\Gamma(X)| \geq |X|$ for any set $X \subseteq V$ or equivalently if G has a (1,2)-factor [2]. clearly, G is a hallian graph if its vertices can be covered by a set of vertex disjoint even paths or odd cycles. A graph G is k-hallian, if for any set A of vertices of order at most k, the subgraph of G induced by the set $V - A$ is hallian. The largest k such that G is k-hallian is called the hallian index of G and is denoted by $h(G)$. Clearly $h(G) \leq \delta(G) - 1$ where $\delta(G)$ denotes the minimum degree among the vertices of G. The middle graph [1] of a graph $G = (V, E)$ denoted by $M(G)$ is a graph with vertex set $V \cup E$, and two vertices in $M(G)$ are adjacent if one is a vertex and other one is an edge incident with it in G or both are adjacent edges in G. The binding number of G is defined by D.R. Woodall [10] as,

$$bind(G) = min_{X \in \sum} \frac{|\Gamma(X)|}{|X|}$$

where $\sum$ is the set of all admissible sets of G and $\Gamma(X) \neq V(G)$. The binding number was intensively studied by [4–6]. If $bind(G)$ is large, then G has edges fairly well distributed. Clearly $bind(G) = 0$ if and only if G has an isolated vertex.

2. Existing Results

We state some existing results without proof that are required for establishing the result in this paper.

Proposition 2.1. [6] If H is a spanning subgraph of G then $bind(H) \leq bind(G)$.

Proposition 2.2. [6] If G has a 1-factor then $bind(G) \geq 1$.

Theorem 2.3. [10] If P_n is a path on n vertices then

$$bind(P_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ \dfrac{n-1}{n+1} & \text{if n is odd} \end{cases}$$

Theorem 2.4. [10] $bind(K_n) = n - 1, n \geq 1$.

Proposition 2.5. [2] For any graph G, $h(G) \leq \delta(G) - 1$.

Theorem 2.6. [2] If G is l- connected and k- hallian, then $|\Gamma(X)| \geq |X| + r$ where $r = min\,\{k, l\}$.

Lemma 2.7. [2] If a graph G on n vertices has $h(G) = \delta(G) - 1$ and $k(G) \geq h(G)$, then $bind(G)$=$\dfrac{n-1}{n-\delta(G)}$.

3. Results

In this section we give the exact values of the binding numbers of middle graphs of some well-known clases of graphs, namely, unicyclic graphs, cycles, paths, complete graphs, complete bipartite graphs.

Proposition 3.1. If G is unicyclic graph then $bind(M(G)) = 1$.

Proof. Let G be a unicyclic graph. Label the vertices and edges of G as $u_1, u_2, \dots, u_n$ and $e_1, e_2, \dots, e_n$ in such a way that u_i is on e_i for $i = 1, 2, \dots, n$. Because of the existance of a one-to-one correspondence between the vertices and edges of G, this labeling is possible. Then by the structure of the middle graph $M(G)$ of G, the existance of 1-factor in $M(G)$, namely the edges of the form $u_i e_i$ for $i = 1, 2, \dots, n$ is evident. Then by proposition 2.2 [6], $bind(M(G)) \geq 1$.

Next choose $X = V(G)$ as asubset of $V(M(G))$ then

$$\frac{|\Gamma_{M(G)}(X)|}{|X|} = 1$$

and thus $bind(M(G)) \leq 1$ and hence the result. ■

Corollary 3.2. $bind(M(C_n)) = 1$.

Proposition 3.3. $bind(M(P_n)) = \dfrac{n-1}{n}$, for $n \geq 1$.

Proof. Label the vertices and edges of P_n as $u_1, u_2, \dots, u_n$ and $e_1, e_2, \dots, e_n$ such that $e_i = u_i u_{i+1}$ for $i = 1, 2, \dots, n-1$. The middle graph $M(P_n)$ of P_n contains P_{2n-1} a path on $2n-1$ vertices as a spanning subgraph. Hence by Proposition 2.1 [6] and Proposition 2.3 [10] we have

$$bind(M(P_n)) \geq bind(P_{2n-1}) = \frac{2n-1-1}{2n-1+1} = \frac{n-1}{n}.$$

Next by choosing $X = \{u_1, u_2, \dots, u_n\}$ a subset of $V(M(P_n))$ we have

$$\Gamma_{M(P_n)}(X) = \{e_1, e_2, \dots, e_{n-1}\},$$

Thus

$$bind(M(P_n)) \leq \frac{|\Gamma_{M(P_n)}(X)|}{|X|} = \frac{n-1}{n}$$

Combining these two we get the required Result. ■

Theorem 3.4.

$$bind(M(K_n)) = \begin{cases} 0 & \text{if } n = 1 \\ 1/2 & \text{if } n = 2 \\ 1 & \text{if } n = 3 \\ \dfrac{n^2+n-2}{n^2-n+2} & \text{if } n \geq 4 \end{cases}$$

This Theorem requires Lemma 3.5 involving use of another graph theoretic parameter called hallian index of a graph introduced by M.Borowiecki and D. Michalak [2].

Lemma 3.5. $h(M(K_n)) = n - 2, n \geq 4$.

Proof. We use induction on n. It is easy to verify that for any set $X \subseteq V(M(K_n))$, $|X| \leq 2$ a graph $M(K_4) - X$ is hallian (i.e., it has (1-2)-factor). Moreover, if we take a set X containing three vertices $e_{i_1}, e_{i_2}, e_{i_3}$ which correspond to the edges of K_4 incident to a vertex u_i; the graph $M(K_4) - X$ is not hallian. Thus $h(M(K_4) = 2$.

Assume that $h(M(K_4) = n - 2$ for any $n \geq 4$. Let us label vertices of $M(K_{n+1})$ in the following way: the vertices of K_{n+1} as $V = u_1, u_2, \ldots, u_{n+1}$ the edges of K_n as $E = e_1, e_2, \ldots, e_{n(n-1)/2}$ and the edges incident to the vertex u_{n+1} as $E' = \acute{e_1}, \acute{e_2}, \ldots, \acute{e_n}$ and let $Y = \{u_{n+1}\} \cup E'$. By the definition of the middle graph we have the following simple observations:

(a) Each vertex u_i together with vertices adjacent to it, induce complete graph on $n+1$ vertices.

(b) Each vertex u_i for $1 \leq i \leq n$ is adjacent to exactly one vertex in the set E' we denote it by $\acute{e_i}$.

(c) Every vertex e_i of E is adjacent to exactly two vertices of E'.

Assume $X \subseteq V(M(K_{n+1}))$, $|X| = n - 1$ and consider two cases: $X \cap Y \neq \phi$ or $X \cap Y = \phi$. In the first case let $X \subseteq V(M(K_n)) \cup Y$, then $M(K_n) - X$ is hallian, by the induction hypothesis. By (a), a graph [Y-X] is complete on atleast two vertices, so it also has an (1-2)-factor. Thus $M(K_{n+1}) - X$ is hallian. In the second case $M(K_n) - X'$ where $|X'| = |X| - 1$, has an (1-2)-factor, by the induction hypothesis.

Let $X' = X - \{x\}$ and F be an (1-2)-factor of $M(K_n) - X'$. If x is contained in an odd cycle of F, then it is obvious that $M(K_n) - X$ has an (1-2)-factor and the set Y is covered by a cycle then in this case $M(K_{n+1}) - X$ is hallian.If x is contained in an even cycle C then we can cover the vertices of $C \cup \{x, y\}$ by an 1-factor. In the case when $y = u_j$, then we cover y by an edge $\left\{y, \acute{e_j}\right\}$ (b) and the vertices of $Y - \{\acute{e}_j\}$, by a cycle, ($Y - \left\{\acute{e_j}\right\}$ induces a complete graph on n vertices), then we have (1-2)-factor of $M(K_{n+1}) - X$. If $y = e_j$ then we can cover y by an edge $\left\{y, \acute{e_i}\right\}$ (c), vertices of $Y - \left\{\acute{e_i}\right\}$ by a cycle, then in this case also $M(K_{n+1}) - X$ is hallian. If x is contained in some $K_2 = \{x, y\}$ of F then we can have (1-2)-factor of $M(K_{n+1}) - X$ in the same way as in above case. Let $X = E'$, then the vertex u_{n+1} is isolated in $M(K_{n+1}) - X$. Thus $M(K_{n+1}) - X$ is not hallian. Finally $h(M(K_{n+1})) = n - 1$. ■

4. Proof of Theorem 3.4

If $n = 1, 2, 3$ the result follows from Theorem 2.4 [10], Theorem 2.3 [10], Corollary 3.2, above respectively. Further we have $h(M(K_n)) = n - 2 = \delta(M(K_n)) - 1$ and $k(M(K_n)) = n - 1$. Thus by Lemma 2.7 [2] and Lemma 3.5, the result follows.

We define the binding number of a middle graph of complete bipartite graph, using the same method as in Theorem 3.4.

Theorem 4.1.

$$bind(M(K_{m,n}) = \begin{cases} \dfrac{n}{n+1} & \text{, if } m = 1, n \geq 2 \\ 1 & \text{, if m=n=1,m=n=2} \\ \dfrac{mn+m+n-1}{mn+n} & , m \geq 2, n \geq 3, m \leq n \end{cases}$$

Lemma 4.2. If $G = M(K_{m,n})$ then $h(G) = m - 1$ for $m \geq 2$, $n \geq 3$ and $m \leq n$.

Proof. If V_1 and V_2 be the partite sets of $K_{m,n}$ with $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. By the structure of G; G contains line graph $L(K_{m,n})$as an induced subgraph.

$L(K_{m,n})$ can be viewed as a cartesian product of K_m and K_n and thus the vertices of $L(K_{m,n})$ can be arranged in m -rows and n columns. After this arrangement the vertices $u_1, u_2, \ldots, u_m$ can be placed in the first column and the vertices $v_1, v_2, \ldots, v_n$ in the last row and now we can add necessary edges so as to form $M(K_{m,n})$. Clearly each row induces K_{n+1} as an induced subgraph except the vertices $v_1, v_2, \ldots, v_n$ in the last row and each column induces K_{m+1} as an induced subgraph except the vertices $u_1, u_2, \ldots, u_m$ in the first column.

Let A be the set of $m - 1$ vertices of G by choosing l_i vertices from i^{th} row where $i = 1, 2, 3, \ldots, m$ and l_j vertices from the last row such that $l_1 + l_2 + \cdots + l_k = m - 1$ and $l_i, l_j \geq 0$. The removal of $l_i$vertices from any row (or column) results in to a induced complete subgraph in the same row (or column) and even paths.

Thus $G - A$ is a hallian and $h(G) \geq m - 1$ but $\delta(G) = m$ so that $h(G) \leq \delta(G) - 1 = m - 1$ which gives us $h(G) = m - 1$. ■

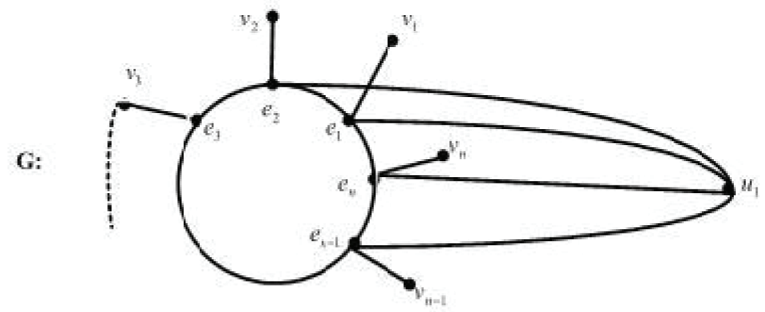
Proposition 4.3. In $M(K_{m,n})$, $|\Gamma(X)| \geq |X| + m - 1$ for every X such that $|\Gamma(X)| \neq V(M(K_{m,n}))$ and $m \leq n$, $m \geq 2$, $n \geq 3$.

Proof. By the structure of $M(K_{m,n})$ it is not difficult to see that removal of any $m - 1$ vertices results into a connected graph and hence $M(K_{m,n})$ is $(m - 1)$. conneced. By the above Lemma $M(K_{m,n})$ is $(m - 1)$-hallian and by the proposition 2.6 [2] $|\Gamma(X)| \geq |X| + m - 1$ holds. ■

5. Proof of the Theorem 4.1

To prove the Theorem we consider three cases.

Case 1:
Let $m = 1$, and $n \geq 2$. By labelling the vertices and edges of $K_{1,n}$ as u_1, to be the center; $v_1, v_2, \ldots, v_n$ as end vertices and $e_1, e_2, \ldots, e_n$ as edges we get $M(K_{1,n})$ as shown in the following figure.
Let $Y \subseteq A \cup B \cup C$ be the admissible set such that Y contains atleast one element of A, B, C and not more than $n - 1$ elements of the form e_i in $M(K_{1,n})$ Otherwise $\Gamma(X) = V(M(K_{1,n}))$ Consider $X_1 = \{v_1, v_2, \ldots, v_n\}$, $X_2 = \{u_1\}$ and $X_3 = \{e_1, e_2, \ldots, e_{n-1}\}$.



1. For $Y \subseteq X_1$, $Y \subseteq X_1 \cup X_2$, $Y \subseteq X_3$ and $Y \subseteq X_1 \cup X_3$ we get respectively $|\Gamma(Y)| = |Y|$,$|\Gamma(Y)| = n$,$|\Gamma(Y)| = n + 1 + |Y|$ and $|\Gamma(Y)| = n$.

2. $Y \subseteq X_1 \cup X_3$ that is $Y = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}, e_{j_1}, e_{j_2}, \ldots, e_{j_l}\}$.

   (a) If $e_{j_1}, e_{j_2}, \ldots, e_{j_l}$ are not incident to any of the $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ Then $|\Gamma(Y)| = n + 1 + n - k = 2n + l - k$, $1 \geq k \leq n - 1$.

   (b) If some v_{i_r}'s are incident with e_{j_r}'s. Without loss of generality that $v_{i_1}, v_{i_2}, \ldots, v_{i_t}$ are incident with $e_{i_1}, e_{i_2}, \ldots, e_{i_t} \in e_{j_1}, e_{j_2}, \ldots, e_{j_l}$ Then $|\Gamma(Y)| = n + l + t$.

3. If $Y \subseteq X_2 \cup X_3$ that is $Y = \{u_1, e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$. Then $|\Gamma(Y)| = n + l + k$.

4. If $Y \subseteq X_1 \cup X_2 \cup X_3$ that is $Y = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}, u_1, e_{j_1}, e_{j_2}, \ldots, e_{j_l}\}$.

   (a) If $e_{j_1}, e_{j_2}, \ldots, e_{j_l}$ are incident with any of $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$. Then $|\Gamma(Y)| = n + l + n - k = 2n + l - k$.

   (b) If some v_{i_r}'s are incident with e_{j_r}'s. Then $|\Gamma(Y)| = n + l + t$. Thus,

$$bind(M(K_{1,n})) = min\left\{1, \frac{n}{n+1}, \frac{n+1+|Y|}{n-1}, \frac{2n+l-k}{k+1}, \frac{n+l+t}{k+1}, \frac{n+1+k}{k+1}\right\}$$

$$= n/n + 1$$

Case 2:
Let $m = 2$ and $n = 1$ then $K_{2,2} = C_4$ and hence by the Carollary 3.2, $bind(M(K_{2,2})) = 1$, $M(K_{1,1}) = P_3$ so by Theorem 2.3 [10], $bind(M(K_{1,1})) = 1/2$

Case 3:
If $m \geq 2$, $n \geq 3$, $m \leq n$, then the result can be proved using Lemma 4.2, Proposition 4.3 ($k(G) \geq h(G)$), Lemma 2.7 [2], the result follows. ■

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