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On Arithmetic Functions

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Abstract

In this paper, we intend to establish some inequalities on Jordan totient function. Using arithmetic functions like the Dedekind's function ψ, the divisors function σ, Euler's totient function φ and their compositions we gave some results on perfect numbers and generalized perfect numbers.

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1. Introduction

Let $\sigma(n)$ denote the sum of divisors of n i.e. $\sigma(n)=\sum_{d|n} d$, where $\sigma(1)=1$. It is well-known that n is said to be a perfect number if $\sigma(n)=2n$. Euclid and Euler showed that even perfect numbers are of the form $n=2^{p-1}(2^p-1)$, where p and 2^p-1 are primes. The prime of the form 2^p-1 is called Mersenne prime and at this moment there are exactly 48 such primes, which mean that there are 48 even perfect numbers [11]. On the other hand, no odd perfect number is known, but no one has yet been able to prove conclusively that none exist. In [5] the author provided detailed information on perfect numbers. Positive integer n with the property $\sigma(n)=2n-1$ is called almost perfect number, while that of $\sigma(n)=2n+1$, quasi-perfect number. For many result and conjectures on this topic, see [4]. D.Suryanarayana [10] introduced the notation of super-perfect numbers. A positive integer n is said to be super- perfect number if $\sigma(\sigma(n))=2n$ and even super-perfect numbers are of the form 2^{p-1} such that 2^p-1 are Mersenne primes. It is asked in this paper and still unsolved whether there are odd super-perfect numbers. Miloni and Bear [8] introduced the concept of

$k-$hyperperfect number and they conjecture that there are $k-$hyperperfect numbers for every $k \in \mathbb{N}$. A. Beg and K. Fogarasi gave a table of $k-$hyperperfect numbers in [1] and defined positive integer n as $k-$hyperperfect number if $\sigma(n) = \frac{k+1}{k} n + \frac{k-1}{k}$ .They proved that if $n = 3^{k-1}(3^k - 2)$, where $3^k - 2$ is prime, then n is a $2-$hyperperfect number. Perfect numbers are $1-$hyper- perfect numbers. If $\sigma(\sigma(n)) = \frac{3}{2} n + \frac{1}{2}$, then n is called super-hyperperfect number. In [1] authors conjectured that all super-hyperperfect numbers are of the form $n = 3^{p-1}$, where p and $\frac{3^p - 1}{2}$ are primes. The Jordan's totient function and Dedekind's function are respectively defined as $J_k(n) = n^k \prod (1 - \frac{1}{p^k})$ and $\psi(n) = n \prod (1 + \frac{1}{p})$, where p runs through the distinct prime divisors of n. Let by convention $J_k(1) = 1$, $\psi(1) = 1$. Euler's totient function is defined as $\varphi(n) = n \prod (1 - \frac{1}{p})$ i.e. $J_1(n) = \varphi(n)$. All these functions are multiplicative, i.e. satisfy the functional equation $f(mn) = f(m) f(n)$ for g.c.d$(m,n) = 1$. In [9] the author defines $\psi \circ \psi$ -perfect, $\psi \circ \sigma$ -perfect, $\psi \circ \varphi$ -perfect numbers, where "$\circ$" denotes composition.

Basic symbols and notations:

$\sigma(n) =$ Sum of divisors of n,

$\psi(n) =$ Dedekind's arithmetic function,

$J_k(n) =$ Jordan totient function,

$\varphi(n) =$ Euler's totient function,

$a/b =$ a divides b,

$a \nmid b =$ a does not divide b,

$pr\{n\} =$ set of distinct prime divisors of n.

Main Results

Proposition2.1. $J_k(ab) \le a^k J_k(b)$, for any $a, b \ge 2$, $k \in \mathbb{N}$, with equality only if, $pr\{a\} \subset pr\{b\}$, where $pr\{a\}$ denotes the set of distinct prime factors of a Proof: Let $ab = \prod_{p|a, p \nmid b} p^{\alpha} . \prod_{q|a, q|b} q^{\beta} . \prod_{r \nmid a, r|b} r^{\gamma}$, then $J_k(ab) = a^k b^k \prod (1 - \frac{1}{p^k}) \prod (1 - \frac{1}{q^k}) \prod (1 - \frac{1}{r^k})$

Since $\prod (1 - \frac{1}{p^k}) < 1$, so $J_k(ab) \le a^k b^k \prod (1 - \frac{1}{q^k}) \prod (1 - \frac{1}{r^k}) = a^k J_k(b)$ Thus, $J_k(ab) \le a^k J_k(b)$, for any $a, b \ge 2$. If $pr\{a\} \subset pr\{b\}$, then $ab = \prod_{p|a, p|b} p^{\alpha} . \prod_{q \nmid a, q|b} q^{\beta}$ and hence $J_k(ab) = a^k J_k(b)$.

Proposition2.2. If $pr\{a\} \not\subset pr\{b\}$, then $J_k(ab) \le (a^k - 1)J_k(b)$ for any $a, b \ge 2$, $k \in$.

Proof: Let $a = \prod p^{\alpha} \prod q^{\beta}$ and $b = \prod p^{\gamma} \prod r^{\lambda}$, where p are the common prime factors, and $q \in pr\{a\}$ are such that $q \notin pr\{b\}$, so $\beta \ge 1$ and $\alpha, \gamma, \lambda \ge 0$ and $pr\{a\} \not\subset pr\{b\}$. $J_k(ab) = a^k b^k \prod (1 - \frac{1}{p^k}) \prod (1 - \frac{1}{q^k}) \prod (1 - \frac{1}{r^k}) = a^k \prod (1 - \frac{1}{q^k}) J_k(b)$

But, $1 - \frac{1}{a^k} = 1 - \frac{1}{(\prod p^{\alpha} \prod q^{\beta})^k} \ge 1 - \frac{1}{\prod q^{k\beta}} \ge 1 - \frac{1}{\prod q^k} \ge \prod (1 - \frac{1}{q^k})$ so, $J_k(ab) = a^k \prod (1 - \frac{1}{q^k}) J_k(b) \le a^k (1 - \frac{1}{a^k}) J_k(b) = (a^k - 1) J_k(b)$ This implies, $J_k(ab) \le (a^k - 1)J_k(b)$, for any $a, b \ge 2$.

Proposition2.3. Let $n \ge 3$, and suppose that n is ψ-deficient number, then $J_k(\psi(n)) < (2^k - 1)n^k$.

Proof: If n is a ψ-deficient number, then $\psi(n) < 2n$. Again for any $n \ge 3$, $\psi(n)$ is an even number. Using Proposition 2.1, we get $J_k(2a) \le a^k J_k(2) = a^k 2^k (1 - \frac{1}{2^k}) = a^k (2^k - 1)$ Let $u = 2a$, an even number, then $J_k(u) \le \frac{u^k}{2^k}(2^k - 1)$. Therefore, $J_k(\psi(n)) \le \frac{\{\psi(n)\}^k}{2^k}(2^k - 1) < \frac{(2n)^k}{2^k}(2^k - 1) = (2^k - 1)n^k$.

Proposition 2.4. For any $n \ge 3$, $J_k(\varphi(n)) < n^k$

Proof: First remark that for any $n \ge 3$, $\varphi(n)$ is even number and $\varphi(n) < n$. Let $\varphi(n) = 2u$, an even integer. Proceeding as Proposition 2.3, we obtain $J_k(2u) \le u^k (2^k - 1)$, i.e. $J_k(\varphi(n)) \le \frac{(2^k - 1)}{2^k}\{\varphi(n)\}^k < \frac{(2^k - 1)}{2^k} n^k < n^k$.

Proposition2.5. If n is an even perfect number, then $J_k(\sigma(n)) < (2^k - 1)n^k$.

Proof: If n is an even perfect number, then $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime and $\sigma(n) = 2n$, so $J_k(\sigma(n)) = j_k(2n) \le n^k j_k(2) = (2^k - 1)n^k$.

Proposition2.6. If n be the product of distinct even perfect numbers $n_1, n_2, ...n_r$, then

a. (a) $\psi(n) = (\frac{2}{3})^{r-1} \psi(n_1)\psi(n_2).....\psi(n_r)$

b. $\varphi(n\sigma(n)) < 2^r n^2$.

c. $\varphi(\psi(n)) = 2^{r-1}\varphi(\psi(n_1))\varphi(\psi(n_2)).....\varphi(\psi(n_r))$

Proof: Let $n = n_1 n_2 ... n_r$, where $n_i = 2^{p_i - 1}(2^{p_i} - 1)$ are distinct perfect numbers with Mersenne primes $2^{p_i} - 1$ (p_i primes, $i = 1, 2, ..., r$)

a. $\psi(n_i) = 3.2^{2p_i - 2}$ for each i, so $\psi(n) = 3.2^{(2p_1 - 1) + (2p_2 - 1) + ...(2p_r - 2)}$ $= (\frac{2}{3})^{r-1}\psi(n_1)\psi(n_2).....\psi(n_r)$.

b. $\varphi(n_i) = 2^{p_i - 1}(2^{p_i - 1} - 1)$ for each i, and clearly $\varphi(n) = 2^{r-1}\varphi(n_1).\varphi(n_2)......\varphi(n_r)$, so $\varphi(n) < 2^{r-1}n$. Therefore $\varphi(n\sigma(n)) \le \varphi(n)\sigma(n) < 2^r n^2$.

$\varphi(\psi(n_i)) = 2^{2(p_i - 1)}$, for each i, so $\varphi(\psi(n)) = \varphi(3.2^{(2p_1 - 1) + (2p_2 - 1) + ...(2p_r - 2)})$

$= 2^{(2p_1 - 1) + (2p_2 - 1) + ...(2p_r - 2)}$

$= 2^{r-1}\varphi(\psi(n_1))\varphi(\psi(n_2)).....\varphi(\psi(n_r))$

Proposition 2.7. If n be the product of distinct 2-hyperperfect numbers $n_1, n_2, ...n_r$, then

a. $\varphi(n) = (\frac{3}{2})^{r-1}\varphi(n_1)\varphi(n_2).....\varphi(n_r)$

b. $\psi(n) = (\frac{3}{4})^{r-1}\psi(n_1)\psi(n_2).....\psi(n_r)$

Proof: Let $n = n_1 n_2 ... n_r$, where $n_i = 3^{k_i - 1}(3^{k_i} - 2)$ distinct 2-hyperperfect numbers with primes are $3^{k_i} - 2$ ($i = 1, 2, 3, ...r$).

a. $\varphi(n_i) = 2.3^{k_i - 1}(3^{k_i - 1} - 1)$, for each i.

Then $\varphi(n) = \frac{2}{3}.3^{(k_1 - 1) + (k_2 - 1) + ...+(k_r - 1)}(3^{k_1 - 1} - 1)(3^{k_2 - 1} - 1)......(3^{k_r - 1} - 1)$ $= (\frac{3}{2})^{r-1}\varphi(n_1)\varphi(n_2)......\varphi(n_r)$ (b) $\psi(n_i) = 2^2 3^{k_i - 2}(3^{k_i} - 1)$, for each i. $\psi(n) = \frac{4}{3}.3^{(k_1 - 1) + (k_2 - 1) + ...+(k_r - 1)}(3^{k_1} - 1)(3^{k_2} - 1)......(3^{k_r} - 1)$ $= (\frac{3}{4})^{r-1}\psi(n_1)\psi(n_2).....\psi(n_r)$.

Proposition2.8. If n is a super-hyperperfect number, then $\varphi(\psi(n)) = \frac{\psi(\varphi(n))}{3}$.

Proof: From definition of super-hyperperfect number we know that $n = 3^{p-1}$, where p and $\frac{3^p - 1}{2}$ are primes. $\psi(n) = 4.3^{p-2}$, so $\varphi(\psi(n)) = 4.3^{p-3} = \frac{4n}{9}$ and $\varphi(n) = 2.3^{p-2}$, so

$\psi(\varphi(n)) = 4.3^{p-2} = \dfrac{4n}{3}$ .Therefore $\varphi(\psi(n)) = \dfrac{\psi(\varphi(n))}{3}$.

Proposition2.9. If n be the product of distinct super-hyperperfect numbers $n_1, n_2, ...n_r$, then

a. $\varphi(\varphi(n)) = \dfrac{3^{r-2}}{2^{r-1}}.\varphi(n_1)\varphi(n_2)......\varphi(n_r)$

b. $\psi(\psi(n)) = \dfrac{3^{r-1}}{2^{2r-3}}.\psi(n_1)\psi(n_2).....\psi(n_r)$

Proof: Let $n = n_1 n_2 ... n_r$, where $n_i = 3^{p_i - 1}$ are distinct super-hyperperfect numbers with p_i and $\dfrac{3^{p_i}-1}{2}$ are primes ($i = 1, 2, 3, ...r.$). Again $\varphi(n_i) = 2.3^{p_i - 2}$ and $\psi(n_i) = 4.3^{p_i - 2}$.

a. $\varphi(n) = \varphi(3^{(p_1-1)+(p_2-1)+......+(p_r-1)}) = \dfrac{2}{3}.3^{(p_1-1)+(p_2-1)+......+(p_r-1)} = 2.3^{(p_1-1)+(p_2-1)+......+(p_r-2)}$, so,

$\varphi(\varphi(n)) = \dfrac{2}{9}.3^{(p_1-1)+(p_2-1)+......+(p_r-1)} = \dfrac{3^{r-2}}{2^{r-1}}.\varphi(n_1)\varphi(n_2)......\varphi(n_r)$.

b. $\psi(n) = \psi(3^{(p_1-1)+(p_2-1)+......(p_r-1)}) = \dfrac{4}{3}.3^{(p_1-1)+(p_2-1)+......(p_r-1)} = 4.3^{(p_1-1)+(p_2-1)+......+(p_r-2)}$, so

$\psi(\psi(n) = \dfrac{8}{3}.3^{(p_1-1)+(p_2-1)+......(p_r-1)} = \dfrac{3^{r-1}}{2^{2r-3}}.\psi(n_1)\psi(n_2).....\psi(n_r)$.

Proposition2.10. (a) There are infinitely many n such that $\psi(J_2(n)) = J_2(\psi(n))$. (b) There are infinitely many n such that $\psi(J_2(n)) = \dfrac{J_2(\psi(n))}{2}$. (c) There are infinitely many n such that $\psi(J_2(n)) < J_2(\psi(n)) < 2n^2$. (d) There are infinitely many n such that $J_2(\psi(n)) < n^2 < \psi(J_2(n))$.

Proof: (a) is valid for $n = 2^a$ for any $a > 1$, since $J_2(2^a) = 3.2^{2a-2}$, $\psi(J_2(2^a)) = 3.2^{2a-1} = \dfrac{3}{2}n^2$ and $\psi(2^a) = 3.2^{a-1}$, $J_2(\psi(2^a)) = 3.2^{2a-1} = \dfrac{3n^2}{2}$. This implies $\psi(J_2(n)) = J_2(\psi(n))$. For (b) put $n = 2^a 3^b$, $(a \geq 1, b \geq 1)$. Then $J_2(2^a 3^b) = 2^{2a+1} 3^{2b-1}$, $\psi(J_2(2^a 3^b)) = \psi(2^{2a+1} 3^{2b-1}) = 2^{2a+2} 3^{2b-1} = \dfrac{4}{3}n^2$. Again $\psi(2^a 3^b) = 2^{a+1} 3^b$, $J_2(\psi(2^a 3^b)) = 2^{2a+3} 3^{2b-1} = \dfrac{8}{3}n^2$, so $\psi(J_2(n)) = \dfrac{J_2(\psi(n))}{2}$. To prove (c) put $n = 2^a 7^b$, $(a, b \geq 1)$ $J_2(2^a 7^b) = 3^2.2^{2a+2} 7^{2b-2}$, $\psi(J_2(2^a 7^b)) = \dfrac{576}{343}n^2$, and $\psi(2^a 7^b) = 3.2^{a+2} 7^{b-1}$,

$J_2(\psi(2^a 7^b)) = \frac{4608}{2401} n^2$, So $\psi(J_2(n)) < J_2(\psi(n)) < 2n^2$.

To prove (d), put $n = 5^a, (a \geq 1)$ $J_2(5^a) = 24.5^{2a-2}, \psi(J_2(5^a)) = \frac{288}{125} n^2$, $\psi(5^a) = 6.5^{a-1}, J_2(\psi(5^a)) = \frac{576}{625} n^2$,

So, $J_2(\psi(n)) < n^2 < \psi(J_2(n))$.

Proposition2.11. (a) There are infinitely many n such that $\varphi(J_2(n)) = J_2(\varphi(n))$

(b) There are infinitely many n such that $J_2(\phi(n)) = \frac{\varphi(J_2(n))}{3}$.

Proof: (a) is valid for $n = 3^a$ for any $a > 1$.

(b) is valid for $n = 2^a 3^b$ for any $a \geq 1, b > 1$. Then an easy computation shows that $J_2(\phi(n)) = \frac{2}{27} n^2$ and $\varphi(J_2(n)) = \frac{2}{9} n^2$, so $J_2(\phi(n)) = \frac{\varphi(J_2(n))}{3}$.

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