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Minimum Unorthodox Measure of Entropy for Prescribed Harmonic Mean and Arithmetic Mean

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Abstract

Maximum entropy probability distribution has been obtained by using different measures. But minimum entropy probability distribution should be obtain for complete information of probability distribution. Entropy is concave function so its minimization is more difficult than maximization. In the present paper, We use Unorthodox measure of entropy to obtain minimum entropy for given Harmonic mean and Arithmetic mean.

Keywords: Unorthodox measure, switching point, consistent values of moments, feasible region.

I INTRODUCTION

The word 'Uncertainty' associated with entropy. Shannon introduced the concept of entropy [7] in 1948 to provide a quantitative measure of this uncertainty. After this measure many other measures of entropy came in existence. These are Renyi's [6], Havrda – Charvat [3] measure etc. Since Shannon entropy is concave function, a lot of work has been done on its maximization and its applications.

Kapur [4] introduced Unorthodox measure of entropy,

$S = -\ln p_{max}$

where $p_{max} = \max\,(p_1, p_{2.----} p_n)$

It is concave function of probability distribution. Entropy is maximum when probability distribution is as equal as possible . Here we have minimum information about system. As we increase information consistent with initial information in the form of moments, entropy decreases. This decreases until we obtain minimum entropy probability distribution. Now, we have complete information about system.

Maximum entropy probability distribution is most unbiased, most uniform and most random while minimum entropy probability distribution is most biased, least uniform and least random. Entropy is concave function so minimization of entropy is complicated than maximization.

Kapur [5] initiated the work to obtain minimum Shannon entropy. Anju Rani [2] obtained minimum entropy for Shannon measure and Havrda- Charvat measure when one moment is prescribed. In this paper, we have obtained analytical expressions for minimum Unorthodox measure of entropy for given Harmonic mean and Arithmetic mean. We have obtained also numerical values of entropy for given these two moments.

II MINIMUM VALUE OF AN UNORTHODOX MEASURE OF ENTROPY WHEN HARMONIC MEAN AND ARITHMETIC MEAN ARE GIVEN: SPECIAL CASE:

Let x be a discrete variate which take all values from 1 to n with probabilities, $p_1, p_{2.----} p_n$. The Harmonic mean and Arithmetic mean of this distribution are prescribed as H and M. There will be many distributions having these particular values of H & M and each of these will have a particular value of entropy. Out of these entropies our aim is to find out minimum value of entropy i.e. S_{min}. Hence, we have to minimize

$$S = -\ln p_{max} \quad -- \quad (1)$$

subject to

$$\sum_{i=1}^{n} p_i = 1 \ , \quad \sum_{i=1}^{n} i\,p_i = M \ , \quad \sum_{i=1}^{n} \frac{p_i}{i} = \frac{1}{H} \quad -- \quad (2)$$

Since there are three linear constraints, the minimum entropy probability distribution will have at most three non-zero components. Let these be p_h, p_k, p_l at points h, k and l respectively, where $1 \leq h < k < l \leq n$.

Then from equation (2)

$$p_h + p_k + p_l = 1 \ , \quad hp_h + kp_k + lp_l = M \ , \quad \frac{p_h}{h} + \frac{p_k}{k} + \frac{p_l}{l} = \frac{1}{H} \quad (3)$$

from equation (3)

$$p_h = \frac{h[kl+H(M-k-l)]}{H(k-h)(l-h)}, \ p_k = \frac{k[H(h+l-M)-hl]}{H(k-h)(l-k)}, p_l = \frac{l[hk+H(M-h-k)]}{H(l-h)(l-k)} \quad (4)$$

Here we study the shifting behavior of p_{max} to calculate minimum entropy. Probability p_h increases with h & k and decreases with l ; p_k decreases with h and increases with k & l ; p_l increases with h, decreases with k, l. According to this we study the shifting of p_{max} from one set of (h, k, l) to another set of (h, k, l).

First we calculate feasible range of M for given value of H. For this we use following expressions by Anju Rani [2].

(i) If H is an integer , then

$$M_{min} = H \quad (5)$$

If H is not an integer, $H = [H] + L$, $0 < L < 1$, where $[H]$ represents integral part of H. Then

$$M_{min} = [H] + \frac{L[H+1]}{[H]+L} \tag{6}$$

(ii) The expression for maximum value of M is given as

$$M_{max} = 1 + n - \frac{n}{H} \tag{7}$$

For the given values of H and M_{min} , probability p_h is zero at point $(1, a, a+1)$ or p_l is zero at point $(a, a+1, n)$ & for the given values of H and M_{max} , probability $p_k = 0$ at point $(1, n-1, n)$. Here $p_h = 0$ for $\{1 \le h < k < H \le l \le n\}$ or $\{1 \le h < k \le H < l \le n\}$ and $p_l = 0$ for $\{1 \le h \le H < k < l \le n\}$ or $\{1 \le h < H \le k < l \le n\}$. For the given values of H and M_{min}, the values of second order moment are same at all existing points and similarly for the given values of H and M_{max} , the values of second order moment are same at all existing points. Here (a, a+1] is interval in which Harmonic mean lies.

Every interval is divided into many subintervals such that for given value of Harmonic mean , the value of minimum entropy for any two subintervals is same. These values are called switching points. For these values, we switch over entropy from one set of values of (h, k, l) to another set of values of (h, k, l).

Let $H \in$ (a, a+1] , $1 \le a < n$, where a is an integer. h can take values 1,2,….,a; k can take values h+1,….,n-1; l can take values a+1,……..,n. We calculate probability distribution in each possible interval for different values of H and M.

For value M_{min}, probability distribution is calculated at point $(a, a+1, n)$ for $a < n-1$ and at point $(1, a, a+1)$ for a = n-1. On the basis of probability distribution, we consider two cases.

CASE I: WHEN $\mathbf{a < n-1}$

There are three possibilities:

(a) When $\mathbf{p_h}$ is maximum probability at point (a, a+1, a+2): In this case, minimum entropy occurs at point p_h(a, a+1, a+2). From equations (1) &(4)

$$s_{min} = -\ln\left[a\left\{\frac{(a+1)(a+2)+H(M-2a-3)}{2H}\right\}\right] \tag{8}$$

(b) When $\mathbf{p_k}$ is maximum probability at point (a, a+1, a+2):

For value M_{min} , $p_k(a, a+1, a+2) = p_l(a-1, a, a+1)$

then p_l(a-1, a, a+1) is considered as maximum probability since p_k is decreasing and p_l is increasing with respect to Arithmetic mean for given value of Harmonic mean.

From equations (1) & (4)

$$s_{min} = -ln\left[(a+1)\left\{\frac{a(a-1)+H(M-2a+1)}{2H}\right\}\right] \tag{9}$$

(c) When $\boldsymbol{p_h(a, a+1, a+2) = p_k(a, a+1, a+2)}$:

In this situation, also p_k(a-1, a, a+1) = p_l(a-1, a, a+1) for M_{min} . Since p_k is decreasing, p_h and p_l are increasing with respect to s[th] order moment , then p_h(a, a+1, a+2) and p_l(a-1, a, a+1) are considered. Out of these values one may be maximum for M, where $M \gtrsim M_{min}$ .Now we take greatest among these probabilities. Hence,

From equation (8)

$$s_{min} = -ln\left[a\left\{\frac{(a+1)(a+2)+H(M-2a-3)}{2H}\right\}\right], \text{ for } p_h > p_l \qquad (10)$$

and, from equation (9)

$$s_{min} = -ln\left[(a+1)\left\{\frac{a(a-1)+H(M-2a+1)}{2H}\right\}\right], \text{for } p_l > p_h \qquad (11)$$

CASE II: WHEN $a = n - 1$

There are two possibilities. These are

(a) If p_k is maximum probability for point (1, a, a+1):

In this case, $s_{min} = -lnp_k(1, a, a+1)$

$$s_{min} = -ln\left[\frac{a\{H(a+2-M)-(a+1)\}}{H(a-1)}\right] \quad \text{(from eq. 4)} \qquad (12)$$

(b) If p_l is maximum probability at point (1, a, a+1):

In this case we consider point (a-1, a, a+1) for entropy to be minimum since p_l is maximum for maximum value of h and

$p_l(1, a, a+1) = p_l(a-1, a, a+1)$, for M_{min}

then, $s_{min} = -lnp_l(a-1, a, a+1)$

$$s_{min} = -ln\left[\frac{(a+1)\{a(a-1)+H(M-2a+1)\}}{2H}\right] \qquad (13)$$

Now, we study three types of points to calculate minimum entropy and its shifting behavior. These are-

(A) p_h at point (a, a+$\propto$, a+$\propto$+1)

(B) p_l at point (a+ β, a+β+1, a+ γ)

(C) p_k at point (1, a+δ, n)

where 1 ≤ $\propto$ ≤ n-2, 1-a ≤ β ≤ n-3, 2 ≤ γ ≤ n-1, 1≤ δ ≤ n-2 .

IF MAXIMUM PROBABILITY OCCURS AT p_h(a, a+$\propto$, a+$\propto$+1):

There are two cases:

(a) a > 1 (b) a=1

(**a) When a > 1:** We study the shifting of minimum entropy from point (a, a+$\propto$, a+$\propto$+1) to another *point*. Maximum probability or minimum entropy may shifts from point p_h(a, a+$\propto$, a+$\propto$+1) to point p_l(a-1, a, a+1). Since p_l is maximum for maximum value of h & minimum value of k and l. Now, we are equating probabilities at two different points to find out switching *point*, at which maximum probability shifts from one point to another point. These probabilities are as:

$$p_h(a, a+\propto, a+\propto+1) = p_l(a-1, a, a+1)$$

$$a\left[\frac{(a+\propto)(a+\propto+1)+H(M-2a-2\propto-1)}{H(\propto)(\propto+1)}\right] = \left[\frac{(a+1)\{a(a-1)+H(M-2a+1)\}}{2H}\right] \text{ (from eq.4)} \tag{14}$$

by solving this equation, we get

$$M_a = \frac{\propto(\propto+1)(a+1)\{a(a-1)-H(2a-1)\}-2a\{(a+\propto)(a+\propto+1)-H(2a+2\propto+1)\}}{H[2a-\propto(\propto+1)(a+1)]} \tag{15}$$

Now, two cases arise:

(A) M_a lies in the feasible region.

(B) M_a does not lie in the feasible region.

(A) $\boldsymbol{M_a}$ lies in the feasible region:

It means, probability distribution should exist for M_a at corresponding points. In this case minimum entropy shifts from p_h(a, a+∝, a+∝+1) to p_l(a-1, a, a+1). So,

$$s_{min} = -ln\left[\frac{a\{(a+\propto)(a+\propto+1)+H(M-2a-2\propto-1)\}}{H(\propto)(\propto+1)}\right], \text{ for } M \le M_a \tag{16}$$

and $$s_{min} = -ln\left[\frac{(a+1)\{a(a-1)+H(M-2a+1)\}}{2H}\right], \text{ for } M_a \le M. \tag{17}$$

(B) If $\boldsymbol{M_a}$ does not lie in the feasible region: In this case maximum probability can not shift from p_h(a, a+∝, a+∝+1) to p_l(a-1, a, a+1). Depending on the value of a, there are two cases:

$(B_1)\ a = 2$ $\qquad$ $(B_2)\ a > 2$

(B$_1$)When $\boldsymbol{a}$ = 2:

Maximum probability shifts to p_l(a-1, a, a+2) since value of h can not be reduced further. Then

$$p_h(a, a+\propto, a+\propto+1) = p_l(a-1, a, a+2) \tag{18}$$

$$a\left[\frac{(a+\propto)(a+\propto+1)+H(M-2a-2\propto-1)}{H\propto(\propto+1)}\right] = (a+2)\left[\frac{a(a-1)+H(M-2a+1)}{6H}\right] \text{ (from eq. 4)}$$

by solving this equation , we get

$$M_b = \frac{\propto(\propto+1)(a+2)\{a(a-1)-H(2a-1)\}-6a\{(a+\propto)(a+\propto+1)-H(2a+2\propto+1)\}}{H[6a-\propto(\propto+1)(a+2)]} \tag{19}$$

If M_b lies in the feasible region then maximum probability shifts from $p_h(a, a+\propto, a+\propto+1)$ to $p_l(a-1, a, a+2)$. Then,

$$s_{min} = -ln\left[a\left\{\frac{(a+\propto)(a+\propto+1)+H(M-2a-2\propto-1)}{H\propto(\propto+1)}\right\}\right], \text{ for } M \le M_b \tag{20}$$

and $$s_{min} = -ln\left[(a+2)\left\{\frac{a(a-1)+H(M-2a+1)}{6H}\right\}\right], \text{ for } M_b \le M \tag{21}$$

If M_b does not lie in the feasible region then value of l is increased gradually upto $a+\propto$ in $p_l(a-1, a, a+2)$ and probabilities will be equated with $p_h(a, a+\propto, a+\propto+1)$ as above. And if any switching point can not be obtained then probabilities are equated as

$$p_h(a, a{+}\propto, a{+}\propto +1) = p_h(a, a{+}\propto +1, a{+}\propto +2). \quad (22)$$

by solving this equation, we get

$$M_c = \frac{H(2a+\propto+1)-a(a+\propto+1)}{H} \quad (23)$$

Minimum entropy shifts from

$p_h(a, a{+}\propto, a{+}\propto +1)$ to $p_h(a, a{+}\propto +1, a{+}\propto +2)$. For $M \leq M_c$

$$s_{min} = -ln\left[a\left\{\frac{(a+\propto)(a+\propto+1)-H(2a+2\propto+1-M)}{H(\propto)(\propto+1)}\right\}\right] \text{ (from eq. 4)} \quad (24)$$

And, for $M_c \leq M$

$$s_{min} = -ln\left[a\left\{\frac{(a+\propto+1)(a+\propto+2)-H(2a+2\propto+3-M)}{H(\propto+1)(\propto+2)}\right\}\right] \text{ (from eq. 4)} \quad (25)$$

(B$_2$) When $a > 2$:

In this case, values of h and k are decreased by one from $p_l(a-1, a, a+1)$ simultaneously and probabilities are equated as:

$$p_h(a, a{+}\propto, a{+}\propto +1) = p_l(a-2, a-1, a+1) \quad (26)$$

$$a\left[\frac{(a+\propto)(a+\propto+1)-H(2a+2\propto+1-M)}{H\propto(\propto+1)}\right] = (a+1)\left[\frac{(a-2)(a-1)+H(M-2a+3)}{6H}\right] \text{(from eq. 4)}$$

by solving this equation, we get

$$M_d = \frac{(a+1)\{(a-2)(a-1)-H(2a-3)\}\propto(\propto+1)-6a\{(a+\propto)(a+\propto+1)-H(2a+2\propto+1)\}}{H[6a-\propto(a+1)(\propto+1)]} \quad (27)$$

If M_d lies in the feasible region then maximum probability shifts from $p_h(a, a{+}\propto, a{+}\propto +1)$ to $p_l(a-2, a-1, a+1)$ and if M_d does not lie in the feasible region then we proceed similarly as above by decreasing the values of h & k from $p_l(a-1, a, a+1)$ and equating probabilities with $p_h(a, a{+}\propto, a{+}\propto +1)$. If any value of switching point can not be obtained then probabilities are equated as:

$$p_h(a, a{+}\propto, a{+}\propto +1) = p_h(a, a{+}\propto +1, a{+}\propto +2).$$

by solving this equation, we get M_c (eq. 23)

For $a{+}\propto +1 = n$, minimum entropy shifts from $p_h(a, a{+}\propto, a{+}\propto +1)$ to $p_k(1, a, n)$ and switching point for this shifting is M_c.

(b)When $\boldsymbol{a = 1}$: In this situation, maximum probability shifts from $p_h(a, a{+}\propto, a{+}\propto +1)$ to $p_h(a, a{+}\propto +1, a{+}\propto +2)$. So,

$$p_h(a, a{+}\propto, a{+}\propto +1) = p_h(a, a{+}\propto +1, a{+}\propto +2) \quad (28)$$

by solving this equation, we get M_c as switching point.

For $a{+}\propto +2 = $n , $p_h(a, a{+}\propto +1, a{+}\propto +2)$ becomes $p_h(1, n-1, n)$ and maximum probability may shifts from $p_h(1, n-1, n)$ to $p_l(1, 2, n)$. For getting switching point for this shifting, probabilities are equated as:

$$p_h(1, n-1, n) = p_l(1, 2, n) \tag{29}$$

by solving this equation, we get

$$M_e = \frac{n(n-3)+H(n+1)}{H(n-1)} \tag{30}$$

$s_{min} = -ln p_h(1, n-1, n)$, for $M \leq M_e$

$s_{min} = -ln p_l(1, 2, n)$, for $M_e \leq M$.

IV IF MAXIMUM PROBABILITY OCCURS AT $p_l(a+\beta, a+\beta+1, a+\gamma)$: There are two cases:

(a) $a+\beta > 1$ (b) $a+\beta = 1$

(a) When $a+\beta > 1$:

From calculation, we observe that maximum probability shifts from $p_l(a+\beta, a+\beta+1, a+\gamma)$ to $p_l(a+\beta-1, a+\beta, a+\gamma)$. So, probabilities are equated as:

$$p_l(a+\beta, a+\beta+1, a+\gamma) = p_l(a+\beta-1, a+\beta, a+\gamma) \tag{31}$$

by solving this equation, we get

$$M_f = \frac{H(2a+\beta+\gamma)-(a+\beta)(a+\gamma)}{H} \tag{32}$$

Now, for $M \leq M_f$

$$s_{min} = -ln\left[(a+\gamma)\left\{\frac{(a+\beta+1)(a+\beta)+H(M-2a-2\beta-1)}{H(\gamma-\beta)(\gamma-\beta-1)}\right\}\right] \text{ (from eq.1 \& 4)} \tag{33}$$

And, for $M_f \leq M$

$$s_{min} = -ln\left[(a+\gamma)\left\{\frac{(a+\beta-1)(a+\beta)+H(M-2a-2\beta+1)}{H(\gamma-\beta)(\gamma-\beta+1)}\right\}\right] \text{ (from eq.1 \&.4)} \tag{34}$$

Similarly, we reduce the values of h and k by one from $p_l(a+\beta-1, a+\beta, a+\gamma)$ simultaneously and find out switching points. We continue the process till $h = 1, k = 2$.

(b) When $a+\beta = 1$:

From calculation, we observe that maximum probability shifts from $p_l(1, 2, a+\gamma)$ to $p_k(1, a+\gamma, n)$. So

$$p_l(1, 2, a+\gamma) = p_k(1, a+\gamma, n) \tag{35}$$

by solving this equation, we get

$$M_g = \frac{H(a+\gamma+1)-(a+\gamma)}{H} \quad (36)$$

Now, for $M \le M_g$

$$s_{min} = -ln\left[(a+\gamma)\left\{\frac{2+H(M-3)}{H(a+\gamma-1)(a+\gamma-2)}\right\}\right] \text{ (from eq. 1 \& 4)} \quad (37)$$

and, for $M_g \le M$

$$s_{min} = -ln\left[(a+\gamma)\left\{\frac{H(n+1-M)-n}{H(a+\gamma-1)(n-a-\gamma)}\right\}\right] \text{ (from eq. 1 \& 4)} \quad (38)$$

V IF MAXIMUM PROBABILITY OCCURS AT $p_k(1, a+\delta, n)$: To obtain switching point for minimum entropy, probabilities are equated as:

$$p_k(1, a+\delta, n) = p_h(a+\delta-1, a+\delta, a+\delta+1) \quad (39)$$

$$\frac{(a+\delta)[H(n+1-M)-n]}{H(a+\delta-1)(n-a-\delta)} = (a+\delta-1)\left[\frac{(a+\delta)(a+\delta+1)+H(M-2a-2\delta-1)}{2H}\right] \text{ (from eq. 4)}$$

by solving this equation, we get

$$M_h = \frac{-(a+\delta-1)^2(n-a-\delta)[(a+\delta)(a+\delta+1)-H(2a+2\delta+1)]+2(a+\delta)\{H(n+1)-n\}}{H[2(a+\delta)+(a+\delta-1)^2(n-a-\delta)]} \quad (40)$$

And $p_k(1, a+\delta, n) = p_l(a+\delta-1, a+\delta, a+\delta+1)$ (41)

$$\frac{(a+\delta)[H(n+1-M)-n]}{H(a+\delta-1)(n-a-\delta)} = (a+\delta+1)\left[\frac{(a+\delta-1)(a+\delta)+H(M-2a-2\delta+1)}{2H}\right] \text{ (from eq. 4)}$$

by solving this equation, we get

$$M_i = \frac{2(a+\delta)\{H(n+1)-n\}-(a+\delta-1)(a+\delta+1)(n-a-\delta)\{(a+\delta-1)(a+\delta)-H(2a+2\delta-1)\}}{H[2(a+\delta)+(a+\delta-1)(a+\delta+1)(n-a-\delta)]} \quad (42)$$

First, we check the existence of probability distributions for values M_h and M_i at corresponding points. If both values lie in the feasible region then minimum value out of M_h and M_i is considered as switching point. If one of these values M_h and M_i lies in the feasible region then feasible value is considered as switching point whether this value is greater or smaller.

If M_h lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_h(a+\delta-1, a+\delta, a+\delta+1)$. So, for $M \le M_h$

$$s_{min} = -ln\left[\frac{(a+\delta)\{H(n+1-M)-n\}}{H(a+\delta-1)(n-a-\delta)}\right] \text{ (from eq. 1 \& 4)} \quad (43)$$

and, for $M_h \le M$

$$s_{min} = -ln\left[\frac{(a+\delta-1)\{(a+\delta)(a+\delta+1)+H(M-2a-2\delta-1)\}}{2H}\right] \text{ (from eq. 1 \& 4)} \quad (44)$$

And, if M_i lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_l(a+\delta-1, a+\delta, a+\delta+1)$.Hence, for $M \leq M_i$

$s_{min} = -ln p_k(1, a+\delta, n)$

and, for $M_i \leq M$

$$s_{min} = -ln\left[(a+\delta+1)\left\{\frac{(a+\delta-1)(a+\delta)+H(M-2a-2\delta+1)}{2H}\right\}\right] \text{ (from eq.1\&4)} \quad (45)$$

If both $\boldsymbol{M_h}$ and $\boldsymbol{M_i}$ do not lie in the feasible region:

In this case, minimum entropy does not shift from p_k $at\ point$ $(1, a+\delta, n)$ to p_h or p_l at point $(a+\delta-1, a+\delta, a+\delta+1)$.

So, three cases arise-

(a) If $a+\delta-1 > 1$, $a+\delta+1 = n$
(b) If $a+\delta-1 = 1$, $a+\delta+1 < n$
(c) If $a+\delta-1 > 1$, $a+\delta+1 < n$

(a) If $\boldsymbol{a+\delta-1 > 1}$, $\boldsymbol{a+\delta+1 = n}$

Since probability p_h is maximum for maximum values of h, k & p_l is maximum for maximum value of h and minimum value of k. Then for equating probabilities, value of h is reduced by one from $p_h(a+\delta-1, a+\delta, a+\delta+1)$ and values of h, k are reduced by one from $p_l(a+\delta-1, a+\delta, a+\delta+1)$. Hence probabilities are equated as:

$$p_k(1, a+\delta, n) = p_h(a+\delta-2, a+\delta, a+\delta+1)$$

$$(a+\delta)\left[\frac{H(n+1-M)-n}{H(a+\delta-1)(n-a-\delta)}\right] = (a+\delta-2)\left[\frac{(a+\delta)(a+\delta+1)+H(M-2a-2\delta-1)}{6H}\right]$$

(from eq.4) - - (46)

by solving this equation, we get

$$M_j = \frac{6(a+\delta)[H(n+1)-n]-(a+\delta-1)(a+\delta-2)(n-a-\delta)[(a+\delta)(a+\delta+1)-H(2a+2\delta+1)]}{H[6(a+\delta)+(a+\delta-1)(a+\delta-2)(n-a-\delta)]} \quad (47)$$

Again, we are equating probabilities as:

$$p_k(1, a+\delta, n) = p_l(a+\delta-2, a+\delta-1, a+\delta+1)$$

$$(a+\delta)\left[\frac{H(n+1-M)-n}{H(a+\delta-1)(n-a-\delta)}\right] = (a+\delta+1)\left[\frac{(a+\delta-2)(a+\delta-1)+H(M-2a-2\delta+3)}{6H}\right]$$

(from eq.4) (48)

by solving this equation, we get

$$M_k = \frac{6(a+\delta)[H(n+1)-n]-(a+\delta+1)(a+\delta-1)(n-a-\delta)[(a+\delta-2)(a+\delta-1)-H(2a+2\delta-3)]}{H[6(a+\delta)+(a+\delta-1)(a+\delta+1)(n-a-\delta)]} \quad (49)$$

If M_j lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_h(a+\delta-2, a+\delta, a+\delta+1)$. For $M \leq M_j$

$$s_{min} = -ln\left[(a+\delta)\left\{\frac{H(n+1-M)-n}{H(a+\delta-1)(n-a-\delta)}\right\}\right] \text{ (from eq. 1\& 4)} \quad (50)$$

and, for $M_j \leq M$

$$s_{min} = -ln\left[(a+\delta-2)\left\{\frac{(a+\delta)(a+\delta+1)+H(M-2a-2\delta-1)}{6H}\right\}\right] \text{ (from eq.1\& 4)} \quad (51)$$

If M_k lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_l(a+\delta-2, a+\delta-1, a+\delta+1)$.

$s_{min} = -lnp_k(1, a+\delta, n)$, for $M \leq M_k$

and, for $M_k \leq M$

$$s_{min} = -ln\left[(a+\delta+1)\left\{\frac{(a+\delta-2)(a+\delta-1)+H(M-2a-2\delta+3)}{6H}\right\}\right] \text{(from eq.1\& 4)} \quad (52)$$

If M_j and M_k do not lie in the feasible region then we proceed similarly and find out switching points by decreasing the values of h & k from p_l upto 1 & 2 respectively and only value of h from p_h up to 1.

(b) If $a+\delta-1 = 1$, $a+\delta+1 < n$

In this case, $h = 1$ is fixed. So there will be no change in value of h. p_h is maximum for maximum value of k & minimum value of l and p_l is maximum for minimum values of k & l then value of k is increased by one and hence value of l is increased by one from $p_h(a+\delta-1, a+\delta, a+\delta+1)$ and value of l is increased by one from $p_l(a+\delta-1, a+\delta, a+\delta+1)$. Hence, we equate probabilities as:

$$p_k(1, a+\delta, n) = p_h(1, a+\delta+1, a+\delta+2)$$

$$(a+\delta)\left[\frac{H(n+1-M)-n}{H(a+\delta-1)(n-a-\delta)}\right] = \left[\frac{(a+\delta+1)(a+\delta+2)+H(M-2a-2\delta-3)}{H(a+\delta)(a+\delta+1)}\right] \text{(from eq.1\& 4)} \quad (53)$$

by solving this equation, we get

$$M_l = \frac{(a+\delta)^2(a+\delta+1)\{H(n+1)-n\}-(a+\delta-1)(n-a-\delta)[(a+\delta+1)(a+\delta+2)-H(2a+2\delta+3)]}{H[(a+\delta)^2(a+\delta+1)+(a+\delta-1)(n-a-\delta)]} \quad (54)$$

Again, we are equating probabilities as:

$$p_k(1, a+\delta, n) = p_l(1, a+\delta, a+\delta+2)$$

$$(a+\delta)\left[\frac{H(n+1-M)-n}{H(a+\delta-1)(n-a-\delta)}\right] = (a+\delta+2)\left[\frac{(a+\delta)+H(M-a-\delta-1)}{2H(a+\delta+1)}\right] \text{(from eq.1\& 4)} \quad (55)$$

by solving this equation, we get

$$M_m = \frac{2(a+\delta+1)(a+\delta)\{H(n+1)-n\}-(a+\delta-1)(a+\delta+2)(n-a-\delta)[(a+\delta)-H(a+\delta+1)]}{H[2(a+\delta+1)(a+\delta)+(a+\delta-1)(a+\delta+2)(n-a-\delta)]} \quad (56)$$

If M_l lies in the feasible region and is minimum then minimum entropy shifts from point $p_k(1, a+\delta, n)$ to $p_h(1, a+\delta+1, a+\delta+2)$. Then, for $M \leq M_l$

$$s_{min} = -ln p_k(1, a+\delta, n)$$

and, for $M_l \leq M$

$$s_{min} = -ln\left[\left\{\frac{(a+\delta+1)(a+\delta+2)+H(M-2a-2\delta-3)}{H(a+\delta)(a+\delta+1)}\right\}\right] \text{ (from eq.1\& 4)} \quad (57)$$

And, if M_m lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_l(1, a+\delta, a+\delta+2)$, then

$s_{min} = -ln p_k(1, a+\delta, n)$, for $M \leq M_m$

and, for $M_m \leq M$

$$s_{min} = -ln\left[(a+\delta+2)\left\{\frac{(a+\delta)+H(M-a-\delta-1)}{2H(a+\delta+1)}\right\}\right] \text{ (from eq.1\& 4)} \quad (58)$$

If both M_l and M_m do not lie in the feasible region then we proceed similarly and increase the value of l upto n and find out switching points.

(c) If $a+\delta-1 > 1$ and $a+\delta+1 < n$:
In this case probabilities are equated as follow:

$$p_k(1, a+\delta, n) = p_h(a+\delta-1, a+\delta+1, a+\delta+2)$$

and $p_k(1, a+\delta, n) = p_l(a+\delta-2, a+\delta-1, a+\delta+1)$

Now,

$$p_k(1, a+\delta, n) = p_h(a+\delta-1, a+\delta+1, a+\delta+2)$$

$$(a+\delta)\left[\frac{H(n+1-M)-n}{H(a+\delta-1)(n-a-\delta)}\right] = (a+\delta-1)\left[\frac{(a+\delta+1)(a+\delta+2)+H(M-2a-2\delta-3)}{6H}\right]$$

(from eq.1&4) (59)

by solving this equation, we get

$$M_n = \frac{6(a+\delta)\{H(n+1)-n\}-(a+\delta-1)^2(n-a-\delta)[(a+\delta+1)(a+\delta+2)-H(2a+2\delta+3)]}{H[6(a+\delta)+(a+\delta-1)^2(n-a-\delta)]} \quad (60)$$

Again, we are equating probabilities as:

$$p_k(1, a+\delta, n) = p_l(a+\delta-2, a+\delta-1, a+\delta+1)$$

$$(a+\delta)\left[\frac{H(n+1-M)-n}{H(a+\delta-1)(n-a-\delta)}\right] =$$

$$(a+\delta+1)\left[\frac{(a+\delta-2)(a+\delta-1)+H(M-2a-2\delta+3)}{6H}\right] \text{ (fromeq.1\&4)} \quad (61)$$

by solving this equation, we get

$$M_o = \frac{6(a+\delta)\{H(n+1)-n\}-(a+\delta+1)(a+\delta-1)(n-a-\delta)[(a+\delta-2)(a+\delta-1)-H(2a+2\delta-3)]}{H[6(a+\delta)+(a+\delta-1)(a+\delta+1)(n-a-\delta)]} \quad (62)$$

If M_n lies in the feasible region and is minimum then minimum entropy shifts from $p_k(1, a+\delta, n)$ to $p_h(a+\delta-1, a+\delta+1, a+\delta+2)$, then

$s_{min} = -lnp_k(1, a+\delta, n)$, for $M \leq M_n$

And, for $M_n \leq M$

$$s_{min} = -ln\left[(a+\delta-1)\left\{\frac{(a+\delta+1)(a+\delta+2)+H(M-2a-2\delta-3)}{6H}\right\}\right] \text{(fromeq.1\&4)} \qquad (63)$$

And if M_o lies in the feasible region and is minimum then minimum entropy shifts to $p_l(a+\delta-2, a+\delta-1, a+\delta+1)$, then

$s_{min} = -lnp_k(1, a+\delta, n)$, for $M \leq M_o$

And, for $M_o \leq$ M

$$s_{min} = -ln\left[(a+\delta+1)\left\{\frac{(a+\delta-2)(a+\delta-1)+H(M-2a-2\delta+3)}{6H}\right\}\right] \text{(from eq.1\& 4)} \qquad (64)$$

If M_n and M_o do not lie in the feasible region then we proceed similarly and increase the values of k & l up to $n-1$ & n in p_h and decrease the values of h & k up to 1 & 2 from p_l respectively.

VI MINIMUM VALUE OF AN UNORHTODOX MEASURE OF ENTROPY WHEN HARMONIC MEAN AND ARITHMETIC MEAN ARE GIVEN: SIX FACED DICE:

Now we calculate minimum value of entropy of an unorthodox measure for six faced dice i.e. $n = 6$. We find out probability distributions for given moments belong to all possible intervals and observe how the maximum probability shifts from one set of points of non-zero probability to another set of points of non-zero probability.

Here we have to minimize

$$s = -lnp_{max} \qquad (65)$$

subject to

$$\sum_{i=1}^{6} p_i = 1 \ , \quad \sum_{i=1}^{6} \frac{p_i}{i} = \frac{1}{H} \ , \quad \sum_{i=1}^{6} i\, p_i = M \qquad (66)$$

CASE - 1 we consider the case when H ∈ (2,3]. In this interval we take values H=2.25, 2.5, 2.75, 3.0. But in the present paper we are considering only for H=2.25. Values of entropies are given in the table 1 for given H=2.25. In this case h=1,2; k=2,3,4,5; l=3,4,5,6.

(a) H=2.25

From equations (6) & (7), $M_{min} = 2.3333$ and $M_{max} = 4.3333$ [$table$ 1].

For M_{min} , p_k is maximum probability. From equation (22),

$[s_{min}]_{p_h(a,a+\propto,a+\propto+1)} = [s_{min}]_{p_h(a,a+\propto+1,a+\propto+2)}$, here a=2, ∝=1, H=2.25.

$[s_{min}]_{p_h(2,3,4)} = [s_{min}]_{p_h(2,4,5)}$

$-ln\left[\frac{3M-5}{3}\right] = -ln\left[\frac{9M-1}{27}\right]$

by solving this equation, we get $M_1 = 2.4444$ [table 1].

The value of M_1 can be obtained from equation (23) for a=2, ∝=1, H=2.25.

$s_{min} = -ln\left[\frac{3M-5}{3}\right]$, for M ∈ [2.3333, 2.4444]

From equation (22),

$[s_{min}]_{p_h(a,a+\propto,a+\propto+1)} = [s_{min}]_{p_h(a,a+\propto+1,a+\propto+2)}$, here a=2, ∝=2, H=2.25.

$[s_{min}]_{p_h(2,4,5)} = [s_{min}]_{p_h(2,5,6)}$

$-ln\left[\frac{9M-1}{27}\right] = -ln\left[\frac{3M+7}{18}\right]$

by solving this equation, we get $M_2 = 2.5556$

The value of M_2 can be obtained from equation (23) for a=2, ∝=2, H=2.25.

$s_{min} = -ln\left[\frac{9M-1}{27}\right]$, for M ∈ [2.4444, 2.5556]

Here $a+\propto+2 = n$, then maximum probability shifts from point $p_h(a, a+\propto+1, a+\propto+2)$ to $p_k(1, a, n)$.

$[s_{min}]_{p_h(a,a+\propto+1,a+\propto+2)} = [s_{min}]_{p_k(1,a,n)}$

$[s_{min}]_{p_h(2,5,6)} = [s_{min}]_{p_k(1,2,6)}$

$-ln\left[\frac{3M+7}{18}\right] = -ln\left[\frac{13-3M}{6}\right]$

by solving this equation, we get $M_3 = 2.6667$

$s_{min} = -ln\left[\frac{3M+7}{18}\right]$, for M ∈ [2.5556, 2.6667]

From equation (55), here a=2, δ=0, H=2.25

$[s_{min}]_{p_k(1,2,6)} = [s_{min}]_{p_l(1,2,4)}$

$-ln\left[\frac{13-3M}{6}\right] = -ln\left[\frac{18M-38}{27}\right]$

by solving this equation, we get $M_4 = 3.0635$

This value can also be calculated from equation (56) for a=2, δ=0, H=2.25, n=6.

$s_{min} = -ln\left[\frac{13-3M}{6}\right]$, for M ∈ [2.6667, 3.0635]

Now, we equate entropies from equation (35) as:

$$[s_{min}]_{p_l(1,2,4)} = [s_{min}]_{p_k(1,4,6)}$$

$$-ln\left[\frac{18M-38}{27}\right] = -ln\left[\frac{78-18M}{27}\right]$$

by solving this equation, we get $M_5 = 3.2222$

Table [1]

M h,k,l	2.3333	2.4444	2.5556	2.6
1,2,3	(0, .6667, .3333)	(.0556, .4444, .50)	(.1111, .2222, .6667)	(.1333, .1333, .7333)
1,2,4		(0, .7778, .2222)	(.037, .6667, .2963)	(.0519, .6222, .3259)
1,2,5			(0, .8148, .1852)	(.0111, .7852, .2037)
1,2,6				
1.3,4				
1.3,5				
1,3,6				
1,4,5				
1,4,6				
1,5,6				
2,3,4	(.6667, .3333, 0)	(.7778, 0, .2222)		
2,3,5	(.6667, .3333, 0)	(.7407, .1667, .0926)	(.8148, 0, .1852)	
2,3,6	(.6667, .3333, 0)	(.7222, .2222, .0556)	(.7778, .1111, .1111)	(.80, .0667, .1333)
2,4,5		(.7778, .2222, 0)	(.8148, 0, .1852)	
2,4,6		(.7778, .2222, 0)	(.8056, .1111, .0833)	(.8167, .0667, .1167)
2.5,6			(.8148, .1852, 0)	(.8222, .1111, .0667)
M h,k,l	2.6667	2.8	3.0	3.2
1,2,3	(.1667, 0, .8333)			
1,2,4	(.0741, .5556, 3704)	(.1185, .4222, .4593)	(.1852, .2222, .5926)	(.2519, .0222, .7259)
1,2,5	(.0278, .7407, .2315)	(.0611, .6519, .287)	(.1111, .5185, .3704)	(.1611, .3852, .4537)
1,2,6	(0, .8333, .1667)	(.0267, .7667,. 2067)	(.0667, .6667, .2667)	(.1067, .5667, .3267)
1.3,4	(.1667, .8333, 0)	(.1889, .6333, .1778)	(.2222, .3333, .4444)	(.2556, .0333, .7111)
1.3,5	(.1667, .8333, 0)	(.1833, .7333, .0833)	(.2083, .5833, .2083)	(.2333, .4333, .3333)
1,3,6	(.1667, .8333, 0)	(.18, .7667, .0533)	(.20, .6667, .1333)	(.22, .5667, .2133)
1,4,5				
1,4,6				
1,5,6				
2,3,4				
2,3,5				
2,3,6	(.8333, 0, .1667)			
2,4,5				
2,4,6	(.8333, 0, .1667)			
2.5,6	(.8333, 0, .1667)			

M h,k,l	3.2222	3.4	3.6	3.7778
1,2,3				
1,2,4	(.2593, 0, .7407)			
1,2,5	(.1667, .3704, .463)	(.2111, .2519, .537)	(.2611, .1185, .6204)	(.3056, 0, .6944)
1,2,6	(.1111, .5556, .3333)	(.1467, .4667, .3867)	(.1867, .3667, .4467)	(.2222, .2778, .50)
1.3,4	(.2593, 0, .7407)			
1.3,5	(.2361, .4167, .3472)	(.2583, .2833, .4583)	(.2833, .1333, .5833)	(.3056, 0, .6944)
1,3,6	(.2222, .5556, .2222)	(.24, .4667, .2933)	(.26, .3667, .3733)	(.2778, .2778, .4444)
1,4,5	(.2593, .7407, 0)	(.2741, .5037, .2222)	(.2907, .237, .4722)	(.3056, 0, .6944)
1,4,6	(.2593, .7407, 0)	(.2711, .6222, .1067)	(.2844, .4889, .2267)	(.2963, .3704, .3333)
				(.3056, .6944, 0)
M h,k,l	3.8	4.0	4.2	4.3333
1,2,3				
1,2,4				
1,2,5				
1,2,6	(.2267, .2667, .5067)	(.2667, .1667, .5667)	(.3067, .0667, .6267)	(.3333, 0, .6667)
1.3,4				
1.3,5				
1,3,6	(.28, .2667, .4533)	(.30, .1667, .5333)	(.32, .0667, .6133)	(.3333, 0, .6667)
1,4,5				
1,4,6	(.2978, .3556, .3467)	(.3111, .2222, .4667)	(.3244, .0889, .5867)	(.3333, 0, .6667)
1,5,6	(.3067, .6667, .0267)	(.3167, .4167, .2667)	(.3267, .1667, .5067)	(.3333, 0, .6667)
2,3,4				
2,3,5				
2,3,6				
2,4,5				
2,4,6				
2.5,6				

$s_{min} = -ln\left[\frac{18M-38}{27}\right]$, for M ∈ [3.0635, 3.2222]

According to section V(c),

$[s_{min}]_{p_k(1,4,6)} = [s_{min}]_{p_l(1,2,5)}$

$-ln\left[\frac{78-18M}{27}\right] = -ln\left[\frac{45M-95}{108}\right]$

by solving this equation, we get $M_6 = 3.4786$

$$s_{min} = -\ln\left[\frac{78-18M}{27}\right], \qquad \text{for } M \in [3.2222, 3.4786]$$

Further, we equate entropies from equation (35) as:

$$[s_{min}]_{p_l(1,2,5)} = [s_{min}]_{p_k(1,5,6)}$$

$$-\ln\left[\frac{45M-95}{108}\right] = -\ln\left[\frac{65-15M}{12}\right]$$

by solving this equation, we get $M_7 = 3.7778$

$$s_{min} = -\ln\left[\frac{45M-95}{108}\right], \qquad \text{for } M \in [3.4786, 3.7778]$$

According to section 4.3.5(a),

$$[s_{min}]_{p_k(1,5,6)} = [s_{min}]_{p_l(1,2,6)}$$

$$-\ln\left[\frac{65-15M}{12}\right] = -\ln\left[\frac{9M-19}{30}\right]$$

by solving this equation, we get $M_8 = 3.9032$

$$s_{min} = -\ln\left[\frac{65-15M}{12}\right], \qquad \text{for } M \in [3.7778, 3.9032]$$

$$s_{min} = -\ln\left[\frac{9M-19}{30}\right], \qquad \text{for } M \in [3.9032, 4.3333]$$

Similarly, we can obtain minimum entropy for all values of Harmonic mean and Arithmetic mean.

VII CONCLUDING REMARKS:

We have obtained the expressions of minimum unorthodox measure of entropy for the given values of Harmonic mean and Arithmetic mean. So, we observe that

For the given values of H and M_{min}, probability distribution is same at all existing points and similarly for the given values of H and M_{max}, probability distribution is same at all existing points.

s_{min} is piecewise concave function of M, for the given value of H.

In a given interval, if p_k is maximum then minimum entropy increases with Arithmetic mean for a given value of Harmonic mean.

If p_h and p_l are maximum probabilities then minimum entropy decreases with Arithmetic mean for a given value of Harmonic mean.

REFERENCES

[1] Anju Rani, Maximum and Minimum value of a moment of a probability distribution when value of some other moment is prescribed, Pure and Applied Mathematical Science 66(1997), 57-67.

[2] Anju Rani, Minimum Entropy Probability Distribution in presence inequalities constraint, Recent Advances in Information Theory. Statistics and Computer Application, Department of Mathematics and Statistics, C.C.S., Haryana Agriculture University, Hisar (1997), 166-170.

[3] J.H. Havrda, & F. Charvat, Quantification Methods of Classificatory Process: Concept of Structural α Entropy, Kybernatica 3(1967), 30-35.

[4] J.N. Kapur, , On minimum entropy probability distribution, MI&TA, JW&S(1994), 203-218.

[5] J.N. Kapur, , An unorthodox measure of entropy, MI&TA, JW&S(1994), 3-11.

[6] A. Renyi, , On measures of information and entropy, Proc. 4th Berkely Symposium Maths Stat. Prob.(1961), 547-561.

[7] C.E. Shannon, , A mathematical theory of communication , Journal of Bell System Tech. 27(1948), 379-423.