Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 4, Number 4 (2012), pp. 465–473

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Cyclic Path Covering Number of Euler Graphs

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Abstract

In this paper, we introduce a technique to find the cyclic path covering number of Euler graphs, by using the t-hypohamiltonian graphs. Also a special type of Euler graph in the name n-gon is also being introduced and a general result of cpcn of n-gon is also found. Finally an algorithm to find the cyclic path covering number of any Euler graph is being developed.

AMS subject classification: 54A40.
Keywords: Hamiltonian graph, T-Hypohamiltonian graph, cyclic cyclomatic graph, cyclic path covering number of graphs, n-gon.

1. Preliminaries

The concept of cyclic path covering is being developed with the concept of mobile traffic in city roads. If we consider the context of mobile traffic streams in a city road network, every vehicle driver would wish to encounter as many green signals at junctions as possible so as to minimize fuel consumption and travel time as also maximize unhindered distance so travelled. With the roads treated as edges and junctions as vertices we treat the whole city roads as a graph G and the Cyclic path covers of G, is an ideal limit for any reasonably realistic measure of "mobility" of traffic flow in the city road network with G as the underlying graph. The concept of path decomposition and path covering number of a graph was introduced by Harary [1]. The preliminary results on this paper were obtained by Harary and Schwenk [2], and Peroche [3]. In this the paths should not intersect any number times and the intersection of any two paths is only vertices of G. To get non-intersecting paths we impose some extra conditions in the definition of Path Covering. All graph considered in this paper are assumed to be connected graphs without isolated points. Let $G = (V, E)$ be a graph. We denote the number of vertices if G by n and the number of edges in G by e.

Definition 1.1. [4] A Cyclic Path covering of a graph G is a collection Γ of paths in G whose union is G satisfying the conditions for distinct paths P_i and P_j with terminal vertices $u, v$and w, z respectively,

$$P_i \cap P_j = \begin{cases} A, & \text{A is the subset of the sets \{u, v, w, z\}} \\ \phi, & \text{if } P_i \text{ and } P_j \text{ are cyclic.} \end{cases}$$

Definition 1.2. [4] The cyclic path covering number γ of G is defined to be the minimum cardinality taken over all cyclic path covers of G. Any cyclic path cover Γ of G with $|\Gamma| = \gamma$ is called a minimum cyclic Path cover of G.

Definition 1.3. Cyclic Cyclomatic Graph [5], [8] The Hamiltonian graph G with unique vertex$v$ with $d(v) = k \geq 2$ such that v is adjacent with exactly $(k-2)$vertices of degree 3 and $(n-k+1)$ vertices of degree 2 is cyclic cyclomatic graph.

Theorem 1.4. Cyclic path covering number γ of union of two Hamiltonian graphs G_1 and G_2, with $G_1 \cap G_2 = \{v_1\}$ whose Cyclic path covering number are $\gamma(G_1)$ and $\gamma(G_2)$ respectively is $\gamma(G_1) + \gamma(G_2) + 1$ and hence $\gamma(G) = \gamma(G_1) + \gamma(G_2) + 1$.

Theorem 1.5. [7] Let G be a unicyclic graph with n vertices of degree 1. Let C be the unique cycle in in G and let m be number of vertices of degree greater than 2 on C. Then

$$\gamma(G) = \begin{cases} 1, & \text{if m =0} \\ n+1, & \text{if m =1} \\ n, & \text{if } m > 1. \end{cases}$$

Theorem 1.6. [7] Cyclic path covering number of union of a Hamiltonian graph G_1 with a vertex v_1 and an edge graph G_2 with edge u_1v_1 is $\gamma(G_1) + 1$.

Corollary 1.7. [7] Let G be the Hamiltonian graph with vertices $v_1, v_2, v_3, \ldots, v_n$. If $G_1 = v_1u_1, G_2 = v_2u_2, \ldots, G_k = v_ku_k$ be the graphs with single edges where $u_i \epsilon G$ for all $i = 1, 2, 3, \ldots, n$. Then $\gamma(G \cup G_1 \cup G_2 \cup \cdots \cup G_k) = \gamma(G) + k$.

Theorem 1.8. [7] Let T_1 and T_2 be any two trees, then $T = T_1 \cup T_2$ will have

(i) $\gamma(T) = \gamma(T_1) + \gamma(T_2) + 1$ if $d(v_1) \neq 1$

(ii) $\gamma(T) = \gamma(T_1) + \gamma(T_2)$ if $d(v_1) \neq 1$ in $T_1$and $d(v_1) = 1$ in T_2.

(iii) $\gamma(T) = \gamma(T_1) + \gamma(T_2) - 1$ if $d(v_1) = 1$.

Theorem 1.9. [7] If G is a Hamiltonian graph and T is a tree with Path covering number $\gamma(G)$ and $\gamma(T)$, then $\gamma(G \cup T) = \gamma(G) + \gamma(T)$ if $d(v_1) = 1$ in T_1, else $\gamma(G \cup T) = \gamma(G) + \gamma(T) + 1$.

Theorem 1.10. If G_1 is any Hamiltonian graph and G_2 is any path of length k with terminal vertices v_1 and v_2 such that $G_1 \cup G_2 = \{v_1, v_2\}$, then $\gamma(G_1 \cup G_2) = \gamma(G_1) + 1$.

Definition 1.11. Graph H_G of G [10] Let G be any graph and H be the sub graph of G. then the sub graph H_G is defined as $H_G = (V(H_G), E(H_G))$, where $E(H_G) = E(G) - E(H)$ and $V(H_G) = (V(G) - V(H)) \cap (V(G) \cup V(H))$.

Definition 1.12. T-Hypo Hamiltonian graph [10] The non-Hamiltonian graph G is said to be T-hypo Hamiltonian graph if for the sub tree T of G, T_G is a maximal Hamiltonian sub graph of G.

Theorem 1.13. [10] For any Hypo hamiltonian graph G with n vertices and e edges, $\gamma(G) = e - n + 1$, if $G - v$ is a Cyclic Cyclomatic Graph, else $\gamma(G) = e - n$.

Theorem 1.14. [10] Let G be a T-hypo Hamiltonian graph then $\gamma(G) = \gamma(T_G) + \gamma(T)$.

2. CPCN of Euler Graph

Theorem 2.1. The cyclic path covering number of Euler graph G is $\gamma(G) = \gamma(G_1) + \gamma(G_2)$, where G_1 is the Hamiltonian graph and $\gamma(G_2)$ is the disjoint union of trees and Hairy Hamiltonian graph.

Proof. Let G be a Euler graph with $G = (n, e)$. Then G may be Hamiltonian. If G is Hamiltonian, then $\gamma\ (G) = e - n + 1$ if G is cyclic cyclomatic, otherwise $e - n$. If G is non Hamiltonian, then G is Euler graph and G has all the vertices with even degree. Let C be a cycle in G with maximum no of vertices. Let G_1 be the sub graph of G such that $G_1 = (V, E)$, $V(G_1) = \{V(C) \cap V(G)\}$ and $E(G_1) = \{e \in E(G)/e = (v_1, v_2)$ such that $v_1, v_2 \in V(G_1)\}$.

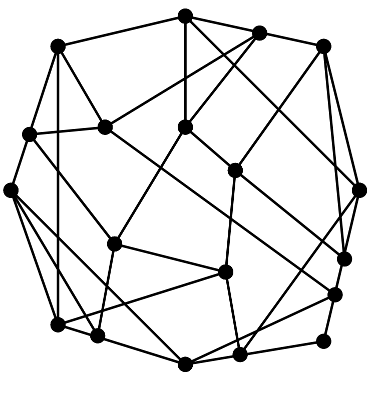

Figure 1: Euler Graph G

Let G_2 be the graph such that ${G_1}_G$. Here the graph G_2 may be connected or disconnected graph. Without loss of generality let us assume G_2 is disconnected graph. Then

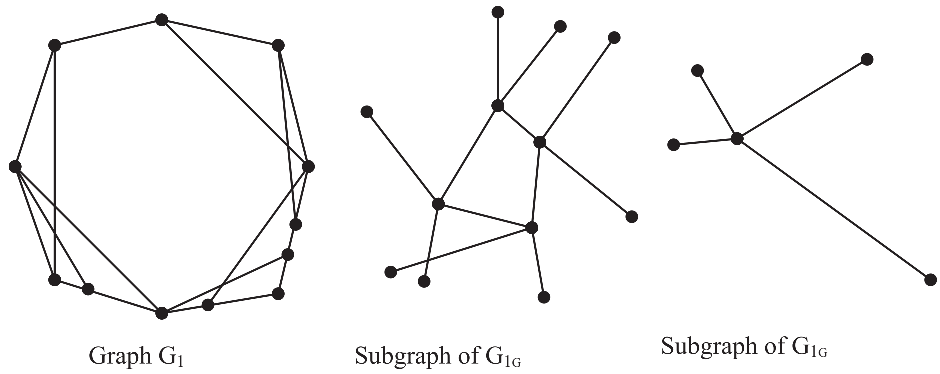

Figure 2: Subgraph G_1 and subgraphs of G_{1G}

each connected subgraph of G_2 may contain tree, Hairy Hamiltonian graph. Then the CPCN of G is found as $\gamma(G) = \gamma(G_1 \cup G_2) = \gamma(G_1) + \gamma(G_2)$. ∎

Theorem 2.2. Let G_1 and G_2 be two simple connected graphs and $G = G_1 \cup G_2$ with $G_1 \cap G_2 = \{v\}$ then $\gamma(G) = \gamma(G_1) + \gamma(G_2) + 1$.

Proof. As G_1 and G_2 are simple connected graph, v is the internal vertex of some path P_1 in Γ_1 and P_2 in Γ_2 of G_2. If Γ is the minimal cpc of G then v cannot be the internal vertex of two paths in Γ. If v is taken as internal vertex of P_1 in Γ then P_2 in Γ_2 is split into two paths P_2^1 and P_2^2 such that $P_2^1 \cup P_2^2 = P_2$. Now, $\Gamma_1 \cup (\Gamma_2 - \{P_2\}) \cup \{P_2^1, P_2^2\}$ is the minimal cpc of G. Thus the number of path covers in Γ is $\gamma(G_1) + \gamma(G_2) - 1 + 2 = \gamma(G_1) + \gamma(G_2) + 1$. ∎

Remark 2.3. The method of finding the Cyclic Path Covering Number of Euler graph is suitable only for smaller graph. For larger graphs which involve more number of vertices like gons this method is not suitable (particularly for the Hamiltonian graphs with one point union which will be discussed in the following section). Also here is a draw back in this method that is finding the circuit C which will include maximum number of vertices. The n-gon graphs are a special type of Euler graph that are defined as follows:

3. n-gon Graphs

Definition 3.1. A one point union of two graphs G_1 and G_2 is defined as $G_1 \cup G_2(v)$, where $v \in V(G_1) \cap V(G_2)$.

Lemma 3.2. The Cyclic Path Covering Number of one point union of a Cycle with any graph G at the point v is $\gamma(G) + 2$, if v is the interior point of some path in the minimal path cover of G, else $\gamma(G) + 1$ if v is the terminal point of some path in the minimal path cover of G.

Proof. Let G be any graph with Cyclic Path Covering Number $\gamma(G)$ and let C be the cycle of length m. Let v be any point such that $G \cap C = \{v\}$. Then the point v may be (i) interior point of any of the path in the minimal Cyclic Path Cover Γ of G or (ii) terminal point of any of the path in the minimal Cyclic Path Cover Γ of G.

Case(i): As v is the interior point of path in minimal path cover of G by definition Cyclic Path Cover v cannot be the interior point of the path unless the cycle C is split into two paths. Therefore C is split into two path P_1 and P_2 which are having v as terminal vertex. Thus v become the terminal vertex of only one path in $G \cup C_m(v)$. Thus $\gamma(G \cup C_m(v)) = \gamma(G) + 2$.

Case(ii): Let P_i be one among the path in Γ which has v as terminal vertex. As C_m is the cycle which starts and ends at v, C_m can be split into as C_{p_1} and C_{p_2} such that $C_m = C_{P_1} \cup C_{P_2}$. Now any one of the path say C_{P_1} can be joined with P_i, such that $P_i \cup C_{P_1}$ is a path. Now the number of paths in the minimal cyclic path cover of $G \cup C_m(v)$ is one path extra than that of $\gamma(G)$. Thus we have $\gamma(G \cup C_m(v)) = \gamma(G) + 1$. ■

Definition 3.3. A n-Gon G_n of level 1 is defined as $\cup_{i=1}^{n}\{C_n \cup C_n(v_i)\}$, Where C_n is the cycle graph with vertices $v_1, v_2, v_3, \cdots v_n$.

Definition 3.4. A 3-gon of level 0, level 1 and level 2 are defined as C_3,

$$\bigcup_{i=1}^{3}\left\{C_3 \bigcup C_3(v_i)\right\}$$

and

$$\bigcup_{1=1}^{3} C_3 \bigcup \left[\bigcup_{i=1}^{3}\left\{C_3 \bigcup C_3(v_i)\right\}\right]$$

and are of the form as in Figure 3.

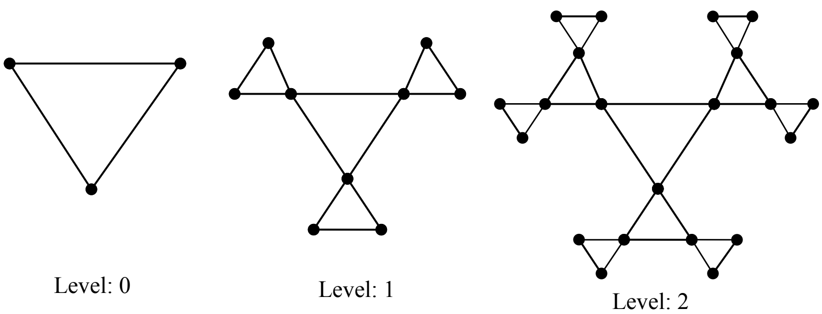

Figure 3: 3-gon of level 0,1 and 2

Definition 3.5. A 4-gon of level 0, level1 and level 2 are defined as C_4,

$$\bigcup_{i=1}^{4}\left\{C_4 \bigcup C_4(v_i)\right\}$$

and

$$\bigcup_{1=1}^{4} C_4 \bigcup \left[\bigcup_{i=1}^{4} \left\{ C_4 \bigcup C_4(v_i) \right\} \right]$$

and are of the form as in Figure 4.

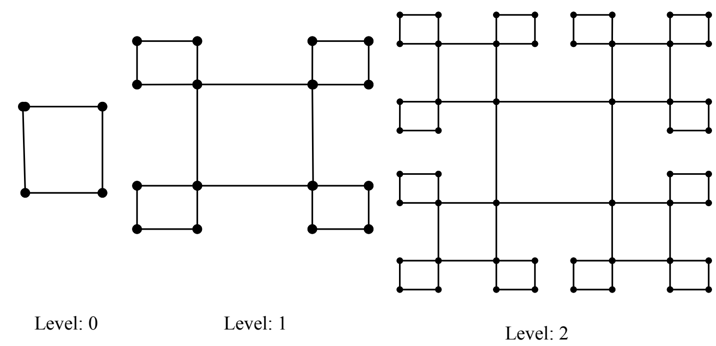

Figure 4: 4-gon of level 0, 1 and 2

Definition 3.6. A 5-gon of level 0, level 1 and level 2 are defined as C_5,

$$\bigcup_{i=1}^{5} \left\{ C_5 \bigcup C_5(v_i) \right\}$$

and

$$\bigcup_{i=1}^{5} C_5 \bigcup \left[\bigcup_{i=1}^{5} \left\{ C_5 \bigcup C_5(v_i) \right\} \right]$$

and are of the form as in Figure: 5.

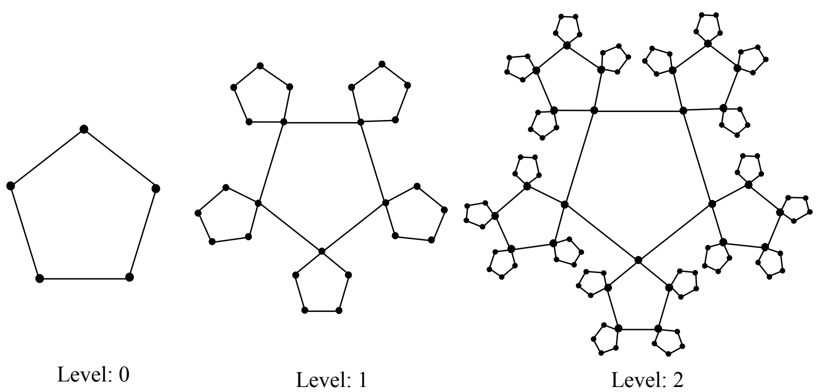

Figure 5: 5-gon of level 0, 1 and 2

Theorem 3.7. The the Cyclic Path Covering number of n-gons is

$$2k + k(2k-3)\left(\frac{(k-1)^{(n-1)} - 1)}{(k-2)}\right), k = 1, 2, 3 \cdots n.$$

Proof. It is obvious that the Cyclic Path Covering Number of any k-gon ($k = 1, 2, 3 \cdots n$) at level 0 is 1. The k-gons at level 0 has cyclic path cover with one path (cycle) with k-1 interior vertices and one terminal vertex.

Level 1: By lemma 3.2 as the one point union of C_k's with k-gon at level 0 gets union at k-1 interior points and at one terminal point the Cyclic Path Covering Number of k-gons at level 1 has $\gamma(k - gons(0)) + (k-1)2 + 1 \times 1 = 1 + 2k - 2 + 1 = 2k$.

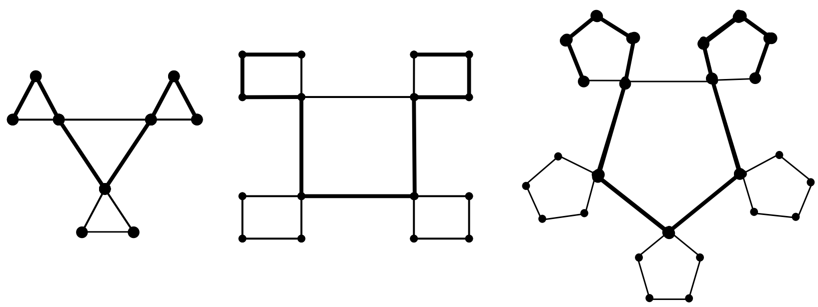

Figure 6: cpc at level 1

Level 2: The one point union of C_k's with k-gon at level 1 gets union at $k(k-2)$ interior points and at k terminal points. Therefore the Cyclic Path Covering Number of k-gons at level 2 is

$$\gamma(k - gons(1)) + k(k-1)2 + k \times 1 = 2k + k(k-2)2 + k = k(2k-1).$$

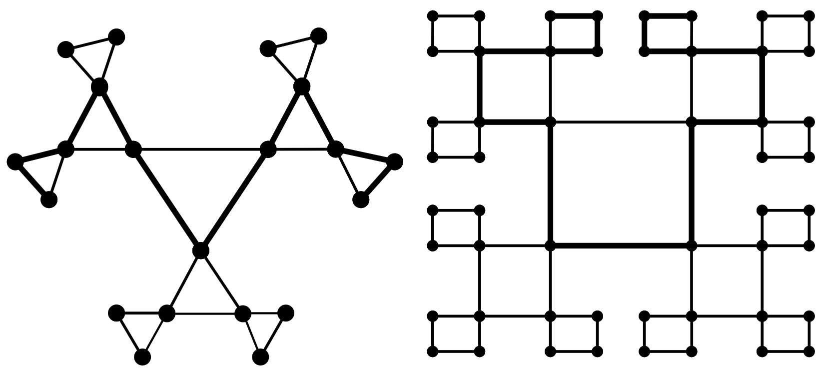

Figure 7: cpc at level 2

Level 3: Similarly the one point union of C_k's with k-gon at level 2 gets union at $k(k-1)(k-2)$ interior points and at $k(k-1)$ terminal points. Therefore the CPCN of k-gons at level 2 is

$$\gamma(k - gons(2)) + k(k-1)(k-2)2 + k \times (k-1) = 2k + k(k-2)2 + k = k(2k^2 - 3k + 2).$$

In the similar way for level 4 we have

$$\begin{aligned}
&\gamma(\texttt{k - gons(3)}) + k(k-1)(k-1)(k-2)2 + k \times (k-1)(k-1) \\
&\quad = 2k + k(k-2)2 + k + k(k-1)(k-2)2 + k(k-1) \\
&\qquad + k(k-1)^2(k-2)2 + k(k-1)^2 \\
&\quad = 2k + k(2k-3)\frac{(k-1)^3 - 1}{(k-2)}.
\end{aligned}$$

In general, for the level n we have for k-gons the Cyclic Path Covering Number is

$$2k + (2k-3)[1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{n-2}] = 2k + (2k-3)k\frac{(k-1)^{n-1} - 1}{k-2}.$$

■

Thus the generalized result is given in Table 1.

Table 1: n-gons with cpcn

Name of the gon	Number of outer gons	γ
3-gon	$3 \times 2^{n-1}$	$6 + 3^2(2^{n-1} - 1)$
4-gon	$4 \times 3^{n-1}$	$8 + 4 \times 5\frac{(3^{n-1} - 1)}{2}$
5-gon	$5 \times 4^{n-1}$	$10 + 5 \times 7\frac{(4^{n-1} - 1)}{3}$
...	...	...
k-gon	$k \times (k-1)^{n-1}$	$2k + (2k-3)k\frac{(k-1)^{n-1} - 1}{k-2}$

Algorithm to find the Cyclic path covering number of Euler Graphs.
The steps are as follows:

i) Split the graph G into edge disjoint subgraphs such that $G = G_1 \cup G_2 \cup \cdots \cup G_k$ where $G_1, G_2, \ldots, G_k$ are Euler graphs, Hamiltonian graphs, T-Hypo hamiltonian graphs and trees.

ii) Find the Cyclic path covering number of $G_1, G_2, \dots, G_k$ using the theorems 1.4 to 1.10, 1.13, 2.1, 2.2 and 3.7.

iii) Now apply the results of the theorems 1.4 to 1.10, 1.13, 2.1, 2.2 and 3.7 to find the Cyclic path covering number of $G_1 \cup G_2 \cup \cdots \cup G_k$.

iv) The resultant value is the $\gamma(G)$.

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