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Retrievability in Bipolar Fuzzy Finite State Machines

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Abstract

In this paper we introduced retrievable, quasi retrivable in bipolar fuzzy finite state machines and discuss their structural properties of bipolar fuzzy finite state machines using the notions of retrivable.

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1. Introduction

Fuzzy sets are kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionstic fuzzy sets, interval-valued fuzzy sets etc. Bipolar-valued fuzzy sets, which are intoduced by Lee [2, 3], are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1].

The theory of fuzzy set was introduced by L.A. Zadeh in 1965 [7]. The mathematical formulation of a fuzzy automaton was first proposed by W.G. Wee in 1967 [6]. E.S. Santos 1968 [5] proposed fuzzy automata as a model of pattern recognition. John N. Mordeson and D.S. Malik gave a detailed account of fuzzy automata and languages in their book 2002 [4].

In [1], Y.B. Jun and J. Kavikumar introduced bipolar fuzzy finite state machines, bipolar successor, bipolar exchange property. In this paper we introduced retrievable, quasi retrivable in bipolar fuzzy finite state machines and discuss their structural properties of bipolar fuzzy finite state machines using the notions of retrivable.

2. Basic Definitions

Definition 2.1. [8] Let X denote a universal set. Then a fuzzy set A in X is set of ordered pairs:

$$A = \{(x,\ \mu_A(x)|x\ \in\ X\},$$

$\mu_A(x)$ is called the membership function or grade of membership of x in A which maps X to the membership space $[0, 1]$.

Definition 2.2. [1] A bipolar-valued fuzzy setφ in X is an object having the form

$$\varphi = \left\{\left(x,\ \varphi^-(x),\ \varphi^+(x)\right) \mid x\ \in\ X\right\}$$

where $\varphi^-\ :\ X\ \to\ [-1,0]$ and $\varphi^+\ :\ X\ \to\ [0,1]$ are the mappings. The positive membership degree $\varphi^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar- valued fuzzy set $\varphi = \left\{(x,\ \varphi^-(x),\ \varphi^+(x))|\ x\ \in\ X\right\}$ and the negative membership degree $\varphi^-(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\varphi = \left\{(x,\ \varphi^-(x),\ \varphi^+(x))|\ x\ \in\ X\right\}$.

Definition 2.3. A finite fuzzy automata is a system of 3 tuples, $M = (Q,\ X,\ f_M)$ where

Q-set of states $\{q_1,\ q_2, \dots,\ q_n\}$
X-alphabets (or) input symbols
f_M-function from $Q \times X \times Q \to [0, 1]$
$f_M(q_i, \sigma, q_j) = \mu\ [0 \le \mu \le 1]$ means when M is in state q_i and reads the input σ will move to the state q_j with weight function μ.

2.1. Bipolar fuzzy finite state machines

Definition 2.4. [1] A bipolar fuzzy finite state machine (bffsm, for short) is a triple $M = (Q, X, \varphi)$, where Q and X are finite nonempty sets, called the set of states and the set of input symbols, respectively and $\varphi = \left\langle\varphi^-, \varphi^+\right\rangle$ is a bipolar fuzzy set in $Q\ \times\ X\ \times Q$.

Let X^* denote the set of all words of elements of X of finite length. Let λ denote the empty word in X^* and $|x|$ denote the length of x for every $x \in X^*$.

Definition 2.5. [1] Let $M = (Q, X, \varphi)$ be a bffsm. Define a bipolar fuzzy set $\varphi_* = \left\langle{\varphi_*}^+, {\varphi_*}^-\right\rangle$ in $Q\ \times\ X^*\ \times\ Q$ by

$$\varphi_*^-(q,\ \lambda,\ p) = \begin{cases} -1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\varphi_*^+(q,\ \lambda,\ p) = \begin{cases} 1 & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\varphi_*^-(q, xa, p) = inf_{r\ \in\ Q}[{\varphi_*}^-(q, x, r) \vee {\varphi_*}^-(r, a, p)]$$

$$\varphi_*{}^{+}(q, xa, p) = sup_{r \in Q}[\varphi_*{}^{+}(q, x, r) \wedge \varphi_*{}^{+}(r, a, p)] \forall\ p, q \in Q,\ x \in X^*$$

and $a \in X$.

Result. Let $M = (Q,\ X,\ \varphi)$ be a bffsm. Then

$$\varphi_*{}^{-}(q, xy, p) = inf_{r \in Q}[\varphi_*{}^{-}(q, x, r) \vee \varphi_*{}^{-}(r, y, p)]$$

$$\varphi_*{}^{+}(q, xy, p) = sup_{r \in Q}[\varphi_*{}^{+}(q, x, r) \wedge \varphi_*{}^{+}(r, y, p)] \forall\ p, q \in Q$$

and $x, y \in X^*$.

2.2. Retrievability in bipolar fuzzy finite state machines

Definition 2.6. **[1]** Let $M = (Q, X, \varphi)$ be a bffsm and let $p, q \in Q$. Then p is called a immediate successor of q if the following condition holds $\exists a \in X$ such that $\varphi_*{}^{-}(q, a, p) < 0$ and $\varphi_*{}^{+}(q, a, p) > 0$. We say that p is a successor of q if the following condition holds $\exists x \in X^*$ such that $\varphi_*{}^{-}(q, x, p) < 0$ and $\varphi_*{}^{+}(q, x, p) > 0$.

We denote by $S(q)$ the set of all successors of q. For any subset T of Q the set of all successors of T denoted by $S(T)$ is defined to be the set $S(T) = \cup\{S(q) \backslash q \in T\}$.

Definition 2.7. **L** et $M = (Q, X, \varphi)$ be a bffsm. M is said to be retrievable if $\forall q \in Q,\ \forall y \in X^*$ if $\exists\ t \in Q$ such that

$$\varphi_*(q, y, t) = \left[\varphi_*{}^{-}(q, y, t) < 0, \varphi_*{}^{+}(q, y, t) > 0\right],$$

then $\exists\ x \in X^*$ such that

$$\varphi_*(t, x, q) = \left[\varphi_*{}^{-}(t, x, q) < 0, \varphi_*{}^{+}(t, x, q) > 0\right].$$

Example

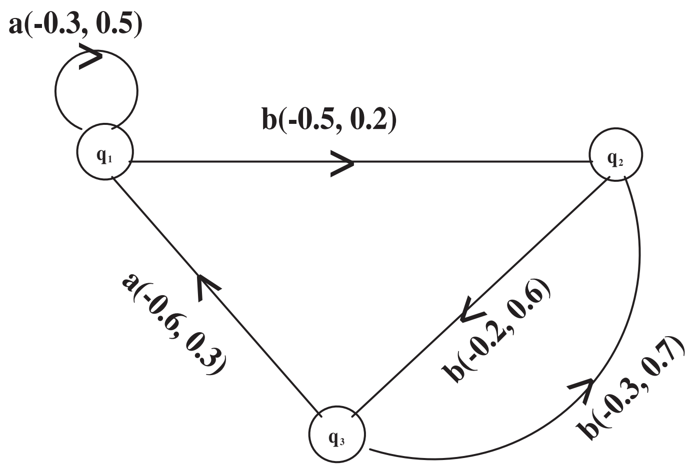

Figure 1.

Definition 2.8. Let $M = (Q, X, \varphi)$ be a bffsm. M is said to be quasi-retrievable if $\forall\, q \in Q, \forall\, y \in X^*$ if $\exists\, t \in Q$ such that

$$\varphi_*(q, y, t) = \left[\varphi_*{}^-(q, y, t) < 0, \varphi_*{}^+(q, y, t) > 0\right],$$

then $\exists\, x \in X^*$ such that

$$\varphi_*(q, yx, q) = \left[\varphi_*{}^-(q, yx, q) < 0, \varphi_*{}^+(q, yx, q) > 0\right].$$

where

$$\varphi_*{}^-(q, yx, q) = inf_{t\, \in\, Q}[\varphi_*{}^-(q, y, t) \vee \varphi_*{}^-(t, x, q)] < 0$$
$$\varphi_*{}^+(q, yx, q) = sup_{t\, \in\, Q}[\varphi_*{}^+(q, y, t) \wedge \varphi_*{}^+(t, x, q)] > 0.$$

Example

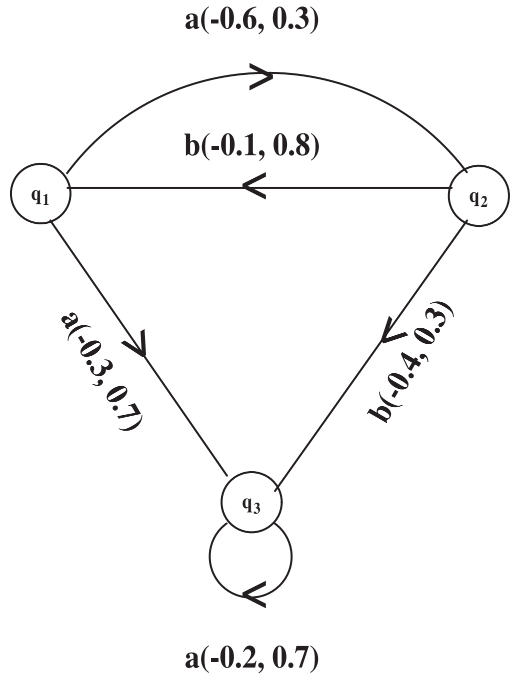

Figure 2.

Definition 2.9. Let $M = (Q,\ X,\ \varphi)$ be a bffsm. Let $q, r, s \in Q$. Then r and s are said to be $q-$ related if $\exists\, y \in X^*$ such that

$$\varphi_*(q, y, r) = \left[\varphi_*{}^-(q, y, r) < 0, \varphi_*{}^+(q, y, r) > 0\right]$$

and

$$\varphi_*(q, y, s) = \left[\varphi_*{}^-(q, y, s) < 0, \varphi_*{}^+(q, y, s) > 0\right].$$

Note. Let r and s are $q-$ related. Then r and s are said to be $q-$ twins if $S(s) = S(r)$.

Definition 2.10. Let $M = (Q, X, \varphi)$ be a bffsm. We say that M satisfies the exchange property if the following condition holds:

Let $p, q \in Q$ and let $T \subseteq Q$. Suppose that if $p \in S(T \cup \{q\})$, $p \notin S(T)$, then $q \in S(T \cup \{p\})$.

Definition 2.11. Let $M = (Q, X, \varphi)$ be a bffsm. Let $T \subseteq Q$. Let ν be a bipolar fuzzy subset of $T \times X \times T$ and let $N = (T, X, \nu)$. The bipolar fuzzy finite state machine N is called a submachine of M if

(i) $\varphi | T \times X \times T = \nu$

(ii) $S_Q(T) \subseteq T$.

Definition 2.12. Let $M = (Q, X, \varphi)$ be a bffsm. Then M is called strongly connected if $\forall p, q \in Q$, $p \in S(q)$.

Definition 2.13. Let $M = (Q, X, \varphi)$ be a bffsm and let $N = (T, X, \nu)$ be a submachine of M. N is called proper if $T \neq Q$ and $T \neq \phi$. If M is strongly connected then M has no proper submachines.

Definition 2.14. Let $M = (Q, X, \varphi)$ be a bffsm. Let $R \subseteq Q$ and $\{N_i = (Q_i, X, \mu_i) \mid i \in I\}$ be the collection of all submachines of M whose state set contains R. Define $< R >= \cap_{i \in I} \{Q_i \mid i \in I\}$. Then $< R >$ is called the submacine generated by R. $< R >$ is the smallest submachine of M whose state set contains R.

Definition 2.15. Let $M = (Q, X, \varphi)$ be a bffsm. M is called singly generated if $\exists q \in Q$ such that $M =< \{q\} >$. In this case q is called a generator of M and we say that M is generated by q.

Definition 2.16. Let $M = (Q, X, \varphi)$ be a bffsm. Let $T \subseteq Q$. If $M =< T >$ then M is generated by T.

Note. Let $M = (Q, X, \varphi)$ be a bffsm. Let $R \subseteq Q$. Then $N = (S(R), X, \varphi_R)$ is a submachine of M where $\varphi_R = \langle \varphi_R^-, \varphi_R^+ \rangle$ where

$$\varphi_R^- = \varphi^- |_{S(R) \times X \times S(R)} < 0$$

$$\varphi_R^+ = \varphi^+ |_{S(R) \times X \times S(R)} > 0$$

Definition 2.17. Let $M = (Q, X, \varphi)$ be a bffsm. Let $T \subseteq Q$. T is called free if $\forall t \in T, t \notin S\left(T \setminus \{t\}\right)$.

Definition 2.18. Let $M = (Q, X, \varphi)$ be a bffsm. Let $T \subseteq Q$. If T is free and $M =< T >$, then T is called a basis of M.

3. Properties of Retrievability in Bipolar Fuzzy Finite State Machines

Lemma 3.1. Let $M = (Q, X, \varphi)$ be a bffsm. Then the following conditions are equivalent.

(i) $\forall\, q, r, p \in Q \forall x, y \in X^*$ if

$$\varphi_*(q, y, r) = \left[{\varphi_*}^-(q, y, r) < 0, {\varphi_*}^+(q, y, r) > 0\right]$$

and

$$\varphi_*(q, yx, p) = \left[{\varphi_*}^-(q, yx, p) < 0, {\varphi_*}^+(q, yx, p) > 0\right]$$

then $p \in S(r)$.

(ii) $\forall q, r, s \in Q$ if r and s are $q-$ related, then r and s are $q-$ twins.

Proof. (i) $\Rightarrow$ (ii) Let $q, r, s \in Q$ be such that r and s are $q-$ related. Then $\exists y \in X^*$ such that

$$\varphi_*(q, y, r) = \left[{\varphi_*}^-(q, y, r) < 0, {\varphi_*}^+(q, y, r) > 0\right]. \tag{1}$$

and

$$\varphi_*(q, y, s) = \left[{\varphi_*}^-(q, y, s) < 0, {\varphi_*}^+(q, y, s) > 0\right].$$

Let $p \in S(s)$ then $\exists x \in X^*$ such that

$$\varphi_*(s, x, p) = \left[{\varphi_*}^-(s, x, p) < 0, {\varphi_*}^+(s, x, p) > 0\right]$$

then by (i)

$$\varphi_*(q, yx, p) = \left[{\varphi_*}^-(q, yx, p) < 0, {\varphi_*}^+(q, yx, p) > 0\right] \tag{2}$$

where

$${\varphi_*}^-(q, yx, p) = inf_{s \in Q}[{\varphi_*}^-(q, y, s) \vee {\varphi_*}^-(s, x, p)] < 0$$

$${\varphi_*}^+(q, yx, p) = sup_{s \in Q}[{\varphi_*}^+(q, y, s) \wedge {\varphi_*}^+(s, x, p)] > 0.$$

From (1) and (2) and by hypothesis $p \in S(r)$ Similarly if $p \in S(r)$ then $p \in S(s)$. Therefore $S(r) = S(s)$. Hence r and s are q-twins.

(ii)$\Rightarrow$(i)

Let $q, r, p \in Q$ and $x, y \in X^*$ be such that

$$\varphi_*(q, y, r) = \left[{\varphi_*}^-(q, y, r) < 0, {\varphi_*}^+(q, y, r) > 0\right]$$

and

$$\varphi_*(q, yx, p) = \left[{\varphi_*}^-(q, yx, p) < 0, {\varphi_*}^+(q, yx, p) > 0\right]$$

where

$$\varphi_*{}^{-}(q, yx, p) = inf_{r \in Q}[\varphi_*{}^{-}(q, y, r) \vee \varphi_*{}^{-}(r, x, p)] < 0$$
$$\varphi_*{}^{+}(q, yx, p) = sup_{r \in Q}[\varphi_*{}^{+}(q, y, r) \wedge \varphi_*{}^{+}(r, x, p)] > 0.$$

Hence $\exists\, s \in Q$ such that

$$\varphi_*(q, y, s) = \left[\varphi_*{}^{-}(q, y, s) < 0, \varphi_*{}^{+}(q, y, s) > 0\right]$$

and

$$\varphi_*(s, x, p) = \left[\varphi_*{}^{-}(s, x, p) < 0, \varphi_*{}^{+}(s, x, p) > 0\right].$$

This implies $p \in S(s)$. Thus by the hypothesis $p \in S(r)$. [Since $S(r) = S(s)$] ■

Lemma 3.2. Let $M = (Q, X, \varphi)$ be a bffsm. Then the following conditions are equivalent.

(i) M is retrievable.

(ii) M is quasi-retrievable and $\forall q, r, s \in Q$ if r and s are $q-$ related then r and s are q-twins.

Proof. (i)$\Rightarrow$ (ii)

It is immediate that retrievability implies quasi retrievability. Since r and s are $q-$ related, there exists $y \in X^*$ such that $\varphi_*^{-}(q, y, r) < 0$ and $\varphi_*^{+}(q, y, r) > 0$. —— (1)
Let $p \in S(s)$. Then therre exists $x \in X^*$ such that $\varphi_*^{-}(s, x, p) < 0$ and $\varphi_*^{+}(s, x, p) > 0$.
Now,$\varphi_*^{-}(q, yx, p) < 0$ and $\varphi_*^{+}(q, yx, p) > 0$.—— (2)
From (1) and (2) and lemma 4.1 $p \in S(r)$. Therefore $S(s) \subseteq S(r)$.——(3)
Similarly we can prove that $S(r) \subseteq S(r)$——(4)
From (3) and (4) $S(s) = S(r)$. Hence r and s are q-twins.

(ii)$\Rightarrow$ (i)

Let $q \in Q$ and $y \in X^*$. Suppose $\exists t \in Q$ such that

$$\varphi_*(q, y, t) = \left[\varphi_*{}^{-}(q, y, t) < 0, \varphi_*{}^{+}(q, y, t) > 0\right].$$

Then $\exists x \in X^*$ such that

$$\varphi_*(q, yx, q) = \left[\varphi_*{}^{-}(q, yx, q) < 0, \varphi_*{}^{+}(q, yx, q) > 0\right].$$

Since M is quasi retrievable by lemma 4.1. $q \in S(t)$. There exist $x \in X^*$ such that $\varphi_*^{-}(t, x, q) < 0$ and $\varphi_*^{+}(t, x, q) > 0$. Hence M is retrievable. ■

Lemma 3.3. Let $M = (Q, X, \varphi)$ be a bffsm. Let $R \subseteq Q$. Then $< R >= (S(R), X, \varphi_R)$.

Proof. Now $< R >= (\cap_{i\in I} Q_i, X, \cap_{i\in I}\varphi_i)$, where $\{N_i \mid i \in I\}$ is the collection of all submachines of M whose state set contains R and $N_i = (Q_i, X, \nu_i), i \in I$. It suffices to show that $S(R) = \cap_{i\in I} Q_i$. Since $(S(R), X, \varphi_R)$ is a submachine of M such that $R \subseteq S(R)$, we have that $\cap_{i\in I} Q_i \subseteq S(R)$. Let $p \in S(R)$. Then $\exists r \in R$ and $x \in X^*$

such that $\varphi_*{}^-(r, x, p) < 0, \varphi_*{}^+(r, x, p) > 0$. Now $r \in \cap_{i\in I}\varphi_i$ and since $< R >$ is a submachine of M, $p \in \cap_{i\in I}\varphi_i$. Thus $S(R) \subseteq \cap_{i\in I}\varphi_i$. Hence $S(R) = \cap_{i\in I}\varphi_i$. ■

Lemma 3.4. Let $M = (Q, X, \varphi)$ be a bffsm. Let $T \subseteq Q$. Then the following conditions are equivalent.

(i) T is a minimal system of generators of M.

(ii) T is a maximally free subset of Q.

(iii) T is a basis of M.

Proof. Let $M = (Q, X, \varphi)$ be a bffsm. Let $T \subseteq Q$.

(i) $\Rightarrow$ (ii)
Let $T = \{p_1, p_2, \ldots, p_n\}$ be a minimal system of generators of M. Let $p_i \in T$. If we assume that $p_i \in S(T/\{p_i\})$, then $M =< T/\{p_i\} >$ which is a contradiction. Therefore $p_i \notin S(T/\{p_i\})$. Hence T is free.

Let $T_1 = \{p_1, p_2, \ldots, p_n, p_{n+1}\}$ be a free subset of Q. $p_{n+1} \in T_1$ then $p_{n+1} \notin S(T_1/p_{n+1})$, $p_{n+1} \notin S(T) =< T >$ which is a contradiction. Hence T is a maximal free subset of Q.

(ii) $\Rightarrow$ (iii)
Let $T = \{p_1, p_2, \ldots, p_n\}$ be a maximal free subset of Q. $< T >\subseteq Q$ ——(1)
Then $p_{n+1} \in Q$ but $p_{n+1} \notin T$. Let $T_1 = \{p_1, p_2, \ldots, p_n, p_{n+1}\}$ is not free, since T is maximal free subset of Q $p_{n+1} \in T_1$ then $p_{n+1} \in S(T_1/\{p_{n+1}\})$, $p_{n+1} \in S(T) =< T >$. Therefore $p_{n+1} \in< T >$——(2)
From (1) and (2) $Q =< T >$. Therefore T is a basis of M.

(iii) $\Rightarrow$ (i)
Since T is a basis of M we have T is a minimal system of generators of M. ■

Note 1. Let $M = (Q,\ X,\ \varphi)$ be a bffsm. Then the following conditions are equivalent.

(i) M satisfies the exchange property

(ii) $\forall p, q \in Q, q \in S(p)$ if and only if $p \in S(q)$.

Lemma 3.5. Let $M = (Q, X, \varphi)$ be a bffsm. Suppose that M satisfies the exchange property. Let $\{q_1, q_2, \ldots, q_n\}$ be a basis of M. Then $M =< q_1 > \cup < q_2 > \cup \cdots \cup < q_n >$.

Proof. We have $< q_i >= (S(q_i), X, \varphi_i)$, where

$$\varphi_i^- = \varphi^-|_{S(q_i)\times X\times S(q_i)} < 0$$

$$\varphi_i^+ = \varphi^+|_{S(q_i)\times X\times S(q_i)} > 0.$$

Now if $i \neq j$, then $S(q_i) \cap S(q_j) = \phi$ since the exchange property is equivalent to the statement that $\forall p, q \in Q, p \in S(q)$ if and only if $q \in S(p)$. Since $M =< q_1, q_2, \ldots, q_n >$, it follows that $M =< q_1 > \cup < q_2 > \cup \cdots \cup < q_n >$. ■

Lemma 3.6. Let $M = (Q, X, \varphi)$ be a bffsm. Then the following conditions are equivalent.

(i) M satisfies the exchange property

(ii) M is the union of strongly connected submachines

(iii) M is retrievable.

Proof. (i) $\Rightarrow$ (ii)
By(i)$M = \cup_{i=1}^{n} < q_i >$, where $\{q_1, q_2, \ldots, q_n\}$ is a basis of M. Also $S(q_i) \cap S(q_j) = \phi$ if $i \neq j$. Let $p, q \in S(q_i)$. Then $p \in S(q_i) \Leftrightarrow q_i \in S(p)$. $q_i \in S(p)$ and $q \in S(q_i)$ $\Rightarrow q \in S(p)$.

Similarly we can show that $p \in S(q)$. Thus $< q_i >$ is strongly connectd. Therefore M is the union of strongly connectd submachines.

(ii) $\Rightarrow$ (iii)
Now, $M = \cup_{i=1}^{n} M_i$, where each $M_i = (Q_i, X, \varphi_i)$ is strongly connected. Let $q \in Q$, $y \in X^*$ such that

$$\varphi_*(q, y, t) = \left[\varphi_*{}^{-}(q, y, t) < 0, \varphi_*{}^{+}(q, y, t) > 0\right]$$

for some $t \in Q$. Now $q \in Q_i$ for some i. Thus $t \in S(q) \subseteq S(Q_i)$. Since M_i is strongly connected, $q \in S(t)$. Hence $\exists x \in X^*$ such that

$$\varphi_*(t, x, q) = \left[\varphi_*{}^{-}(t, x, q) < 0, \varphi_*{}^{+}(t, x, q) > 0\right].$$

Thus M is retrievable.

(iii) $\Rightarrow$ (i)
Let $p \in Q$. By hypothesis, $\forall y \in X^*$ if $\exists t \in Q$ such that

$$\varphi_*(p, y, t) = \left\langle\varphi_*{}^{-}(p, y, t) < 0, \varphi_*{}^{+}(p, y, t) > 0\right\rangle, \tag{1}$$

then $\exists x \in X^*$ such that

$$\varphi_*(t, x, p) = \left\langle\varphi_*{}^{-}(t, x, p) < 0, \varphi_*{}^{+}(t, x, p) > 0\right\rangle. \tag{2}$$

From (1) $\Rightarrow$ $t \in S(p)$ From (2) $\Rightarrow$ $p \in S(t)$. Hence M satisfies the exchange property. ■

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