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A Remark on “The $3n + 1$-problem and Holomorphic Dynamics”

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Abstract

The $3x + 1$ function defined on the partitions of odd integers of the form 4n+1 or 4n+3,n is a positive integer is $F : Z^+ \to Z^+$ given by

$$F(n) = \begin{cases} 3\left(\dfrac{n-1}{4}\right) + 1 \cong \dfrac{3n+1}{4} & if\ n \equiv 1 mod 4 \\ 3\left(\dfrac{n-1}{4}\right) + 2 \cong \dfrac{3n+1}{2} & if\ n \equiv -1 mod 4 \\ \dfrac{n}{2} & if\ n\ is\ even \end{cases}$$

and it is conjectured that every positive integer will be eventually periodic,and the cycle it fall onto is $1 \to 2 \to 1$.The entire holomorphic function interpolating this $3x + 1$ function is constructed and its holomorphic dynamics and the dynamics on the reals are discussed.

AMS subject classification: 11Y55.
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1. Introduction

The well known $3n+1$ problem is the successive iteration of the function $f : Z^+ \to Z^+$ defined by

$$f(n) = \begin{cases} 3n+1 & if\ n\ is\ odd \\ \frac{n}{2} & if\ n \text{ is even} \end{cases} \tag{1}$$

for any positive integer n, eventually reaches (1).

Simon Letherman et al. [7], interpreted this $3n+1$ problem as an iterative procedure and constructed a holomorphic map $f^* : C \to C$ given by

$$f^*(z) = \frac{z}{2} + \frac{1}{2}(1 - cos\pi z) + \frac{1}{\pi}\left(\frac{1}{2} - cos\pi z\right) sin\pi z + h(z)sin^2\pi z \tag{2}$$

where h is any entire holomorphic function. This f^* preserves the reals or $f^*(\bar{z}) = \overline{f^*(z)}$ for all z. From the point of view of holomorphic dynamics, he showed that the existence of any integer that is not eventually periodic implies the existence of a "Wandering domain" for the entire holomorphic map (2). On integers, the holomorphic map (2) agrees with the iteration function (1) and hence the problem can equivalently be formulated as "Iterating the function f^* on any positive integer will land at the number 1 after finitely many steps".

(3)

Now we consider $O = 2N - 1$, the set of odd positive integers and it is partitioned into the residue classes (a) and (b) as follows:

$$\begin{aligned} (a) &= \{n_0 \in O : \frac{n_0+1}{2} \text{ is odd}\} = \{1\ mod\ 4\} \\ (b) &= \{n_0 \in O : \frac{n_0+1}{2} \text{ is even}\} = \{-1\ mod\ 4\} \quad [2] \end{aligned} \tag{4}$$

Given an odd integer n_0, the iterate of n_0, $f(n_0) = 3n_0 + 1$ is an even integer.In order to apply successive iterations, we rewrite $f(n_0)$ as $f(n_0) = 3(n_0 - 1) + 4$ where n_0 is odd. If n_0 is in (a), then $n_0 - 1$ is divisible by at least 4 and hence $\frac{3(n_0-1)}{4} + 1 = \frac{f(n_0)}{4} = f^3(n_0)$, which we denote as $G(n_0)$. Similarly if n_0 is in (b), then $n_0 - 1$ is divisible by at least 2 and hence $\frac{3(n_0-1)}{2} + 2 = \frac{f(n_0)}{2} = f^3(n_0)$, which we denote as $H(n_0)$. Hence we have $F : Z^+ \to Z^+$ given by

$$F(n) = \begin{cases} 3\left(\frac{n-1}{4}\right) + 1 \cong \frac{3n+1}{4} & if\ n\ is\ in\ (a) \\ 3\left(\frac{n-1}{4}\right) + 2 \cong \frac{3n+1}{2} & if\ n\ is\ in\ (b) \\ \frac{n}{2} & if\ n\ is\ even \end{cases} \tag{5}$$

and for any positive integer n, the successive iterations of F eventually reaches 1. Note that, for any odd integer n_0 in (a), that is $n_0 \equiv 1 \ mod \ 4$, $G(n_0)$ is always of the form $3k+1$ which is either odd or even.

Also $k = \dfrac{G(n_0)-1}{3}$ is odd if $G(n_0)$ is even and hence $G\left(\dfrac{G(n_0)-1}{3}\right) = F^2(G(n_0))$ and $H\left(\dfrac{G(n_0)-1}{3}\right) = F(G(n_0))$.

Interpreting (5) as an iterative procedure and we consider a holomorphic map $F^* : C \to C$ given by

$$F^*(z) = \frac{z}{4}(1+cos\pi z) + \frac{3z+1}{16}\left\{(1-cos\pi z)\left(3 - sin\frac{\pi z}{2}\right)\right\}$$
$$+ \frac{3}{16\pi}\left\{(1-cos\pi z)\left(3-sin\frac{\pi z}{2}\right)sin\pi z\right\} - \frac{1}{4\pi}(1+cos\pi z)sin\pi z \qquad (6)$$

This function F^* preserves reals which happens if and only if $F^*(\bar{z}) = (\overline{F^*(z)})$ for all z.

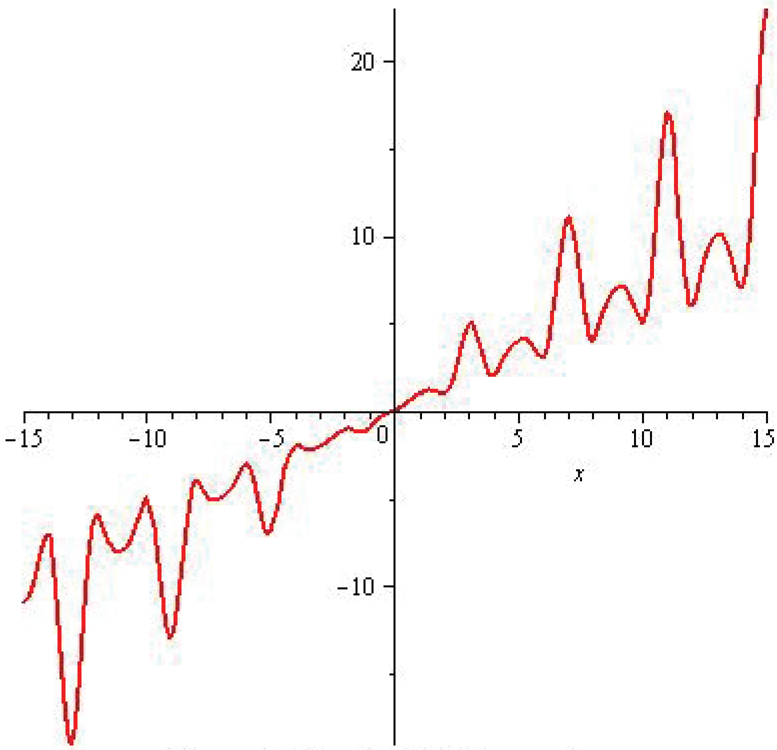

Figure 1: Graph of F* (z) on reals

As

$$sin\frac{n\pi}{2} = \begin{cases} 1 \ \ if \ n \ is \ in \ (a) \\ -1 \ \ if \ n \ is \ in \ (b) \\ 0 \ \ if \ n \ is \ even \end{cases}$$

and

$$cos\frac{n\pi}{2}=\begin{cases}0 \ \ if\ n\ is\ odd\\(-1)^{\frac{n}{2}}\ \ if\ n\ is\ even\end{cases}$$

we have

$$F^{*}(n)=\begin{cases}\dfrac{3n+1}{4}\ \ if\ n\ is\ in\ (a)\\ \dfrac{3n+1}{2}\ \ if\ n\ is\ in\ (b)\\ \dfrac{n}{2}\ \ if\ n\ is\ even\end{cases}\quad .$$

That is our function F^{*} agrees on all integers with the iteration function F and hence the problem can equivalently be formulated, similar to (3) thus:

Conjecture 1.1. (The holomorphic $3n+1$ problem) Iterating the function F^{*} on any positive integer will land at the number 1 after finitely many steps.

The idea of the paper is to present some observations related to our holomorphic map F^{*} defined in (6) which we observe in the sections viz "The holomorphic $3n+1$ map" and "The dynamics on the real line", in [7].

2. The holomorphic $3n+1$ problem

We recall all the fundamental concepts on holomorphic dynamics discussed in section 2 viz "Holomorphic Dynamics" of [7]. Now we begin to find the critical points of F^{*} and ivestigating types of Fatou sets that can contain integers. To get the critical points of F^{*}, we have

$$\begin{aligned}F^{*'}(z)=sin^{2}\pi z\left\{\frac{13}{8}-\frac{15}{32}sin\frac{\pi z}{2}\right\}\\ +sin\pi z\left\{\left(\frac{5z+3}{16}\right)\pi-\frac{3\pi}{2}\left(\frac{3z+1}{16}\right)sin\frac{\pi z}{2}\right\}.\end{aligned}\tag{7}$$

Observe that all integers are critical points of F^{*}.

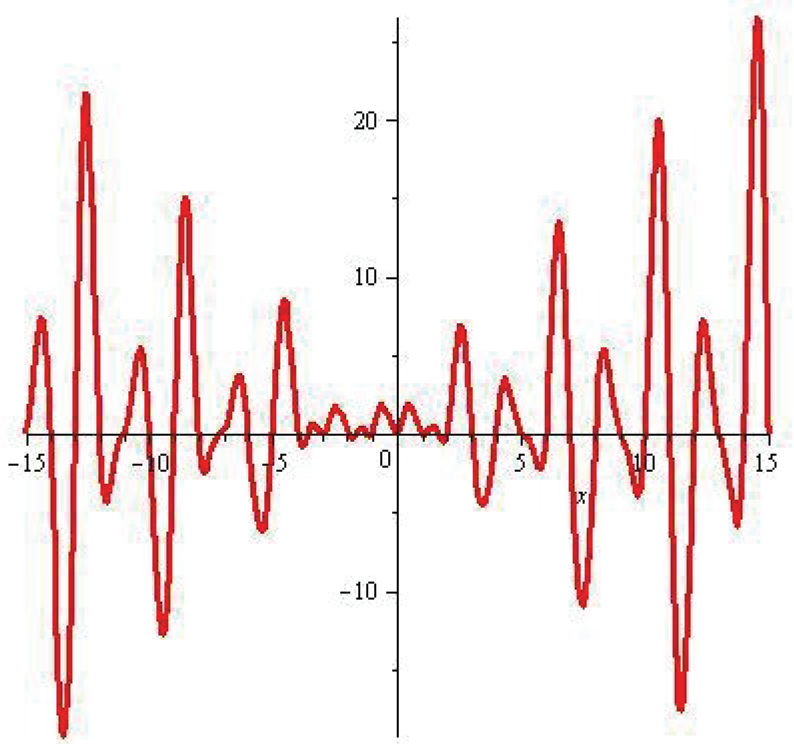

Figure 2 : Graph of F*'(z) on reals

We have the following lemmas and conjecture relating to holomorphic map F^* which are similar to lemmas 3.1 and 3.2 of in section 3 of [7].

Lemma 2.1. (Interpolating the $3n+1$ problem) Any entire holomorphic map that interpolates the $3n+1$ problem in such a way that all the integers are critcal points, is of the form (6).

Also the conjecture can then equivalently be formulated as, "Every positive integer is in the basin of attraction of the super attracting fixed point 1".

Lemma 2.2. (Integers in Fatou set) All the integers are in the Fatou set of F^*.

Proof. The integers 0 and 1 are the super attracting fixed points and thus in the Fatou set. Using the same argument discussed in Lemma 3.2 of [7], we have for $z = n + \delta$ in the open neighbourhood $U_n = \left\{ z \in C : |z-n| < \frac{1}{|2\pi^2 n|} \right\}$; we have $|sin\pi z| = |sin\pi\delta| \leq sinh\pi|\delta|$ and

$$\left|sin\frac{\pi}{2}z\right| = \begin{cases} \left|cos\frac{\pi}{2}\delta\right| \leq cosh\frac{\pi}{2}|\delta| \;\; if\ n\ is\ even \\ \left|sin\frac{\pi}{2}\delta\right| \leq sinh\frac{\pi}{2}|\delta| \;\; if\ n\ is\ odd \end{cases}$$

For $|\delta| < \frac{1}{|2\pi^2 n|} \leq \frac{1}{|2\pi^2|}$, we have $sinh\pi|\delta| \leq 1.005\pi|\delta|$;

$$sinh\frac{\pi}{2}|\delta| \leq 1.001\frac{\pi}{2}|\delta|$$

and

$$cosh\frac{\pi}{2}|\delta| \leq 12.606\frac{\pi}{2}|\delta|.$$

Thus

$$|F^{*'}(z)| \leq \left|sin^2\pi z\right|\left|\frac{13}{8} - \frac{15}{32}sin\frac{\pi}{2}z\right| + |\pi sin\pi z|\left|\frac{5z+3}{16} - \frac{3}{2}\left(\frac{3z+1}{16}\right)sin\frac{\pi z}{2}\right|$$

When n is odd,

$$\begin{aligned}
|F^{*'}(z)| &\leq |sin^2\pi\delta|\left|\frac{13}{8} - \frac{15}{32}\right|sin\frac{\pi}{2}\delta|| \\
&\quad + |\pi sin\pi\delta||\frac{5(n+\delta)+3}{16} - \frac{3}{2}\left(\frac{3(n+\delta)+1}{16}\right)|sin\frac{\pi}{2}\delta|| \\
\Rightarrow |F^{*'}(z)| &\leq (1.005)^2\pi^2|\delta|^2\left|\frac{13}{8} - \frac{15}{32}(1.001)\frac{\pi}{2}\right|\delta|| + (\pi^2(1.005)|\delta|)|\frac{5(n+\delta)+3}{16} \\
&\quad - \frac{3}{2}\left(\frac{3(n+\delta)+1}{16}\right)(1.001)\frac{\pi}{2}|\delta|| \\
&\leq \frac{(1.005)^2}{4}\left|\frac{13}{8} - \frac{15}{32}\frac{1.001}{4\pi}\right| + \frac{1.005}{2n}\left|\frac{5(n+\delta)+3}{16}\right. \\
&\quad \left. - \frac{3}{2}\frac{(3(n+\delta)+1)}{16}(1.001)\frac{\pi}{2}|\delta|\right| \\
&\leq \frac{(1.005)^2}{4}\left|\frac{13}{8} - \frac{15}{32}\frac{1.001}{4\pi}\right| + \frac{1.005}{2}\left|\left(\frac{5}{16} + \frac{5}{32\pi^2} + \frac{3}{16}\right)\right. \\
&\quad \left. - \frac{3}{2}\left(\frac{3}{16} + \frac{3}{32\pi^2} + \frac{1}{16}\right)\left(\frac{1.001}{4\pi}\right)\right| \\
&\leq 0.644519474
\end{aligned}$$

When n is even,

$$
\begin{aligned}
|F^{*'}(z)| &\leq |sin^2\pi\delta||\frac{13}{8} - \frac{15}{32}|cos\frac{\pi}{2}\delta|| + |\pi sin\pi\delta||\frac{5(n+\delta)+3}{16} \\
&\quad - \frac{3}{2}\left(\frac{3(n+\delta)+1}{16}\right)|cos\frac{\pi}{2}|\delta|| \\
&\leq (1.005)^2\pi^2|\delta|^2|\frac{13}{8} - \frac{15}{32}(12.606)\frac{\pi}{2}|\delta|| \\
&\quad + (\pi^2(1.005)|\delta|)|\frac{5(n+\delta)+3}{16} - \frac{3}{2}\left(\frac{3(n+\delta)+1}{16}\right)(12.606)\frac{\pi}{2}|\delta|| \\
&\leq \frac{(1.005)^2}{4}|\frac{13}{8} - \frac{15}{32}\frac{(12.606)}{4\pi}| + \frac{1.005}{2n}|\frac{5(n+\delta)+3}{16} \\
&\quad - \frac{3}{2}\frac{(3(n+\delta)+1)}{16}(12.606)\frac{\pi}{2}|\delta| \\
&\leq \frac{(1.005)^2}{4}|\frac{13}{8} - \frac{15}{32}\frac{(12.606)}{4\pi}| + \frac{1.005}{2}|\left(\frac{5}{16} + \frac{5}{32\pi^2} + \frac{3}{16}\right) \\
&\quad - \frac{3}{2}\left(\frac{3}{16} + \frac{3}{32\pi^2} + \frac{1}{16}\right)\left(\frac{12.606}{4\pi}\right)| \\
&\leq 0.156693406
\end{aligned}
$$

Hence we have $f(U_0) \subset U_{f(0)}$ and $f(U_n) \subset U_{f(n)}$, which proves the lemma.

Using the same principles, concepts and arguments used for proving the lemmas, proposition, corollaries in section 3 viz "The holomorphic $3n + 1$ map" of [7], we have similar lemmas, proposition, corollaries relating to our holomorphic map F^*.

Lemma 2.3. (Integers in Fatou Set) If a Fatou component of F^* corresponding to an attracting orbit contains an integer, then this orbit is super attracting.

No Fatou component corresponding to a rationally indifferent orbit or to a Siegel disk can contain an integer. If F^* preserves the reals, then on sigel disk can intersect the real axis.

Proposition 2.4. Every Fatou component of the map F^* is simply connected, whether or not F^* preserves the reals.

Corollary 2.5. If F^* preserves the reals, no domain at infinity can intersect the real line. In particular, no integer can be in a domain at infinity.

Proposition 2.6. (No integers in domains at infinity) No domains at infinity can contain the integer, whether or not F^* preserves the reals.

Conjecture 2.7. (No wandering domains at integers) The holomorphic map F^* contains all the integers in its Fatou set and has no simply connected wandering domain intersecting the integers.

3. Dynamics on the real line

In this section we have some observations relating to our holomorphic map F^* while discussing the dynamics of F^* restricted to the real line, which have a slight difference with the dynamics of maps f^* in (2) restricted to the real line in section 4 of [7].

Lemma 3.1. (Wandering real numbers) If F^* is a real continuous interpolation of $3n + 1$ problem then between any pair of consecutive positive non zero even integer and an odd integer which is congruent to -1 *mod* 4, both the integers are greater than 1, there is a cantor set of points that diverge to ∞ strictly monotonically.

Proof. For any integer $n \geq 3$, the real interval $[4n + 2, 4n + 3]$ has sub intervals that map onto $[4n + 6, 4n + 7]$ and onto $[4n + 10, 4n + 11]$. For any one-sided sequences of integers in $\{1, 2\}$ there is thus at least one real number in $[4n + 2, 4n + 3]$, that diverges to infinity strictly monotonically such that its orbit is restricted to the intervals of the type $[4n_i + 2, 4n_i + 3]$ with integers n_i such that $n_{i+1} - n_i$ is the prescribed sequence of integers. This is a cantor set: A non empty compact completely disconnected set without isolated points. Infact, for n sufficiently large, the derivatives at these intervals will be strictly larger than 1, and we obtain exactly a cantor set. Also the interval $[2, 3]$ has a subinterval covering $[3, 4]$.

Any interval $[4n + 1, 4n]$ with $n \geq 2$ will cover the interval $[4n + 2, 4n + 3]$ in an orientation reversing way. Moreover, for our map F^*, the interval $[-2, -1]$ maps to -1 which is a repelling fixed point. ■

Lemma 3.2. If F^* is a real continuous interpolation of $3n+1$ problem then between any pair of consecutive positive non zero even integer and an odd integer which is congruent to 1 *mod* 4, both the integers are greater than 1, there is a cantor set of points that diverge to $+\infty$ strictly monotonically.

Proof. For any integer $n \geq 3$, the real interval $[4n, 4n + 1]$ has sub intervals that map onto $[4n - 4, 4n - 3]$ and onto $[4n - 8, 4n - 7]$ for $n \geq 5$ and onto $[4n - 12, 4n - 11]$ for $n \geq 8$ etc. For any one-sided sequences of integers $\{1, 2\}$, there is thus at least one real number in $[4n, 4n + 1]$ that diverges to infinity strictly monotonically such that its orbit is restricted to intervals of the type $[4n_i, 4n_i + 1]$ with integers n_i such that $n_{i+1} - n_i$ is the prescribed sequence of integers. This is a cantor set. Also any interval $[4n + 1, 4n + 2]$ with $n \geq 3$ will cover the interval $[4n - 4, 4n - 3]$, $[4n - 8, 4n - 7]$ in an orientation reversing way. Moreover for the map F^*, the interval $[1, 2]$ is mapped to 1 which is a repelling fixed point. ■

Lemma 3.3. (Julia set between integers) For the given real continuous interpolation of the $3n + 1$ problem, there is a real fixed point between any pair of cosecutive non zero even integer and an odd integer which is congruent to -1 *mod* 4 except possibly $\{-2,-1\}$, $\{-1,0\}$.

In the holomorphic case when F^* preserves the reals, then between any pair of consecutive even integers which contains an odd integer, congruent to -1 *mod* 4, there is a fixed point in the Julia set and there is a point in the Julia set between any pair of consecutive non zero even integer and an odd integer which is congruent to -1 *mod* 4 except possibly $\{-1,-2\}$.

Proof. We have $|f(n)| < |n|$ for non zero even integers and for odd integers which are congruent to 1 *mod* 4 except 1, while $|f(n)| > |n|$ for odd integers which are congruent to -1 *mod* 4 except -1. Now the intermediate value theorem yields the existence of a fixed point between any pair of cosecutive non zero even integer and an odd integer which is congruent to -1 *mod* 4 except for the pair $\{-1,-1\}$, $\{-1,0\}$.

Between any non zero even integer and the adjacent odd integer, congruent to -1 *mod* 4 with greater absolute value, the graph of a real holomorphic $f(z)$ has to cross the graph of the identity from below and the derivative at such a fixed point has to be at least 1. Such fixed points are thus repelling or rationally indifferent and hence in the Julia set. For similar reasons, there are fixed points in the Julia set between the super attracting fixed points 0 and -1.

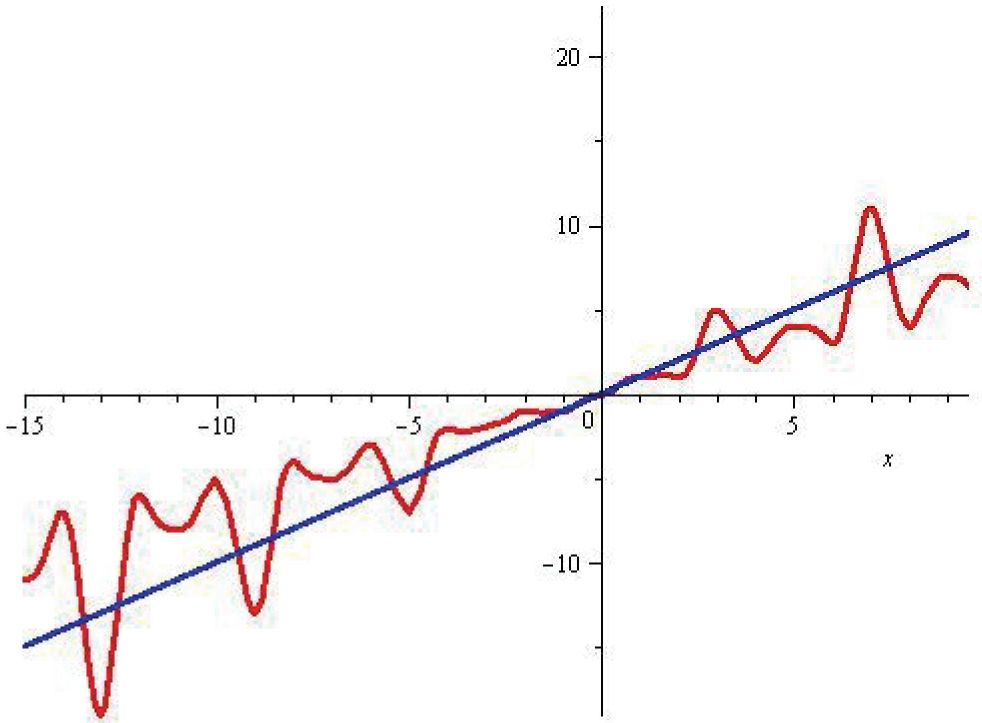

Figure 3: Graph of intersection of F*(x) and y = x

Consider the case of two adjacent integers such that the odd one has smaller absolute value: $\pm 4n$ and $\pm(4n-1)$. For $n \geq 2$, such an interval maps over two adjacent even integers, contains an odd integer which is congruent to -1 *mod* 4 and there is a point in the Julia set in between. Also there is a fixed point in the Julia set between the super attracting fixed point 0 and -1. Finally, $\{-1, -2\}$, both map to -1 and could be in the same Fatou component. ■

Observation 3.4. For the given real continuous interpolation of the $3n + 1$ problem, there is no real fixed point between any pair of consecutive non zero even integer and an odd integer which is congruent to 1 *mod* 4,(except possibly $\{0, 1\}$ and $\{1, 2\}$).

Observation 3.5. In the holomorphic case if F^* preserves the reals, then between any pair of consecutive even integers which contains an odd integer which is congruent to 1 *mod* 4, there is no fixed point in the Julia set.

Lemma 3.6. There is a point in the Julia set between any pair of consecutive non zero even integer and an odd integer which is congruent to 1 *mod* 4, except possibly $\{1, 2\}$.

Proof. For $n \geq 2$, $[4n + 1, 4n + 2]$ and $[4n, 4n + 1]$ for $n \geq 1$, both map over two adjacent even integers, which contains an odd integer, congruent to -1 *mod* 4, there is a point in the Julia set. As $\{1, 2\}$ both map to 1 and could be in the same Fatou component. ■

Conjecture 3.7. If the holomorphic map F^* preserves the reals, then no two integers are in the same Fatou component, except possibly for -1, -2, 1 and 2.

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