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On von Neumann's Inequality for Complex Trigonometric Polynomials of Several Variables

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Abstract

We prove that, in general, von Neumann's inequality cannot hold up to some constant when complex trigonometric polynomials over $\mathbb{T}^n$ are allowed.

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1. Introduction and Main Result

In 1951, von Neumann [8] showed that for any contraction linear operator T on a Hilbert space the following inequality:

$$\|p(T)\| \leq \|p\|_\infty ,$$

holds for all complex polynomials $p(z)$ over the unit disk, where $\|p\|_\infty$ denotes the supremum norm of p over the unit disk. This result was generalised by many people. In particular, Brehmer [2] proved in 1961 that von Neumann's inequality also holds for families $\{S_1, S_2, \ldots, S_n\}$ of commuting operators on a complex Hilbert space with

$$\sum_{i=1}^{n} \|S_i\|^2 \leq 1.$$

In 1963, Ando [1] established the natural generalisation of von Neumann's inequality for polynomials in two commuting contractions. In 1974, Varopoulos [7] proved that the analogue of von Neumann's inequality fails for 3 or more commuting contractions.

There are several such counterexamples in the literature [6]. In 1978, Lubin [5] proved that if $T_1, \dots, T_n$ are commuting contractions on a Hilbert space, then

$$\|p(T_1, \dots, T_n)\| \leq sup\left\{|p(z_1, \dots, z_n)| : |z_i| < \sqrt{n}\right\}, \tag{1}$$

for any polynomial $p(z_1, \dots, z_n)$ over $\mathbb{D}^n$.

This result, which has many engineering applications, is a fundamental tool in operator theory [4, 6].

In this paper, we are mainly concerned with the following question: Does von Neumann's inequality hold up to some constant for the case where complex trigonometric polynomials over $\mathbb{T}^n$ are allowed? From complex analytic trigonometric polynomials over $\mathbb{T}^n$ which do not satisfy von Neumann's inequality, it is possible to construct sequences of positive trigonometric polynomials over $\mathbb{T}^n$ in which von Neumann's inequality cannot hold up to some constant for n-tuples of commuting contractions:

Theorem 1.1. In general, von Neumann's inequality cannot hold up to some constant when complex trigonometric polynomials over $\mathbb{T}^n$ are allowed.

2. Proof of the Main Result

In this section, we show that the failure of von Neumann's inequality for 3-tuples of commuting contractions allows us to construct a sequence of positive trigonometric polynomials over $\mathbb{T}^3$ in which von Neumann's inequality cannot hold up to some constant for 3-tuples of commuting contractions. Let $\mathcal{Q}_n$ denote the algebra of complex trigonometric polynomials over $\mathbb{T}^n$. Recall that if

$$f(z_1, \dots, z_n) = \sum_{(k_1, \dots, k_n) \in S} \hat{f}(k_1, \dots, k_n) z_1^{k_1} \dots z_n^{k_n},$$

is a complex trigonometric polynomial over $\mathbb{T}^n$, then

$$\|f\|_\infty = sup\{|f(z_1, \dots, z_n)| : |z_1| = \dots = |z_n| = 1\}.$$

Proof of the Theorem 1.1

Let us show that, for the case where complex trigonometric polynomials are allowed, in general, there exists a set $\{T_1, \dots, T_n\}$ of commuting contractions on a Hilbert space such that

$$sup\left\{\frac{\|f(T_1, \dots, T_n)\|}{\|f\|_\infty} : f \neq 0, f \in \mathcal{Q}_n\right\} = +\infty.$$

First of all, from complex analytic trigonometric polynomials over $\mathbb{T}^3$ which do not satisfy von Neumann's inequality, we can show that for $n = 3$, the above statement is true. That is, when complex trigonometric polynomials over $\mathbb{T}^3$ are allowed, there exists a set $\{T_1, T_2, T_3\}$ of commuting contractions on a Hilbert space such that

$$sup\left\{\frac{\|f(T_1,\dots,T_3)\|}{\|f\|_\infty} : f \neq 0, f \in \mathcal{Q}_3\right\} = +\infty.$$

Let us consider the following well known homogeneous polynomial over $\mathbb{D}^3$ introduced by Kaijser-Varopoulos [6]:

$$g(z_1,z_2,z_3) = z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3.$$

Let us set

$$f(z_1,z_2,z_3) = g(z_1,z_2,z_3)|\mathbb{T}^3 = z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3.$$

The polynomial $f(z_1,z_2,z_3)$ is a complex trigonometric polynomial on $\mathbb{T}^3$. It is well known that

$$\|f\|_\infty = sup\,\{|f(z_1,z_2,z_3)| : |z_i| = 1\} = 5.$$

It is clear that

$$\left\|f^*f\right\|_\infty = \|f\|_\infty^2 = 25.$$

Let us consider the following well known 3-tuple of commuting contractions $\{T_1, T_2, T_3\}$ on $\mathbb{C}^5$ defined by setting

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 \end{pmatrix}$$

and

$$T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}.$$

A simple calculation shows that

$$(T_1)^2 = (T_2)^2 = (T_3)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_1T_2 = T_1T_3 = T_2T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{3}} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently,

$$f(T_1, T_2, T_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{9}{\sqrt{3}} & 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \|f(T_1, T_2, T_3)\| = 3\sqrt{3}.$$

From this we can conclude that

$$|f(T_1, T_2, T_3)|^2 = f(T_1, T_2, T_3)^* f(T_1, T_2, T_3) = \begin{pmatrix} 27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us point out that the structure of this matrix allows us to generate a sequence of positive trigonometric polynomials over $\mathbb{T}^3$ which do not satisfy von Neumann's inequality. Indeed, let us define the sequence $\{\varphi_k(z_1, z_2, z_3)\}_{k=0}^{\infty}$ of complex trigonometric polynomials over $\mathbb{T}^3$ by setting

$$\varphi_0(z_1, z_2, z_3) = f(z_1, z_2, z_3)^* f(z_1, z_2, z_3) = |f(z_1, z_2, z_3)|^2$$

and

$$\varphi_k(z_1, z_2, z_3) = \varphi_{k-1}(z_1, z_2, z_3)^* \varphi_{k-1}(z_1.z_2, z_3) = |\varphi_{k-1}(z_1, z_2, z_3)|^2 .$$

Trigonometric polynomials of this sequence do not satisfy von Neumann's inequality. In fact, let us estimate $\|\varphi_k(T_1, T_2, T_3)\|$, $\|\varphi_k\|_\infty$. A simple calculation shows that

$$\|\varphi_k\|_\infty = (25)^{2^k} , \|\varphi_k\|_\infty = \left\|(\varphi_{k-1})^* \left(\varphi_{k-1}\right)\right\|_\infty = \|\varphi_{k-1}\|_\infty^2 ,$$

$$\varphi_0(T_1, T_2, T_3) = \begin{pmatrix} 27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\varphi_k(T_1, T_2, T_3) = (\varphi_0(T_1, T_2, T_3))^{2^k} = \begin{pmatrix} (27)^{2^k} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to see that

$$\frac{\|\varphi_k(T_1, T_2, T_3)\|}{\|\varphi_k\|_\infty} = \frac{(27)^{2^k}}{(25)^{2^k}} = \left(\frac{27}{25}\right)^{2^k}.$$

It is very simple to observe that

$$\|\varphi_k\|_\infty < \|\varphi_k(T_1, T_2, T_3)\|.$$

The sequence $\{\varphi_k(z_1, z_2, z_3)\}_{k=0}^{\infty}$ also satisfies

$$\sup\left\{\frac{\|\varphi_k(T_1, T_2, T_3)\|}{\|\varphi_k\|_\infty} : k \in \mathbb{N}\right\} = +\infty.$$

Indeed,

$$\sup\left\{\frac{\|\varphi_k(T_1, T_2, T_3)\|}{\|\varphi_k\|_\infty} : k \in \mathbb{N}\right\} = \sup\left\{\left(\frac{27}{25}\right)^{2^k} : k \in \mathbb{N}\right\} = +\infty.$$

Recall that we are considering here only the case where complex trigonometric polynomials are allowed. We know that

$$\{\varphi_k(z_1, z_2, z_3)\}_{k=0}^{\infty} \subseteq \mathcal{Q}_3,$$

therefore,

$$\sup\left\{\frac{\|\varphi_k(T_1, T_2, T_3)\|}{\|\varphi_k\|_\infty} : k \in \mathbb{N}\right\} \leq \sup\left\{\frac{\|f(T_1, T_2, T_3)\|}{\|f\|_\infty} : f \neq 0, f \in \mathcal{Q}_3\right\}.$$

Consequently,

$$\sup\left\{\frac{\|f(T_1, T_2, T_3)\|}{\|f\|_\infty} : f \neq 0, f \in \mathcal{Q}_3\right\} = +\infty,$$

because

$$\sup\left\{\frac{\|\varphi_k(T_1, T_2, T_3)\|}{\|\varphi_k\|_\infty} : k \in \mathbb{N}\right\} = +\infty.$$

This yields us the desired result. ■

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