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Modular Identities and Explicit Values of a New Continued Fraction of Ramanujan

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Abstract

In this paper, we study a new continued fraction of Ramanujan and find its modular identities and some explicit values.

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1. Introduction

Throughout the paper, we assume $|q| < 1$. As usual, for positive integers n and any complex number a, we write

$$(a)_n := (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n). \tag{1}$$

Ramanujan's theta-functions are defined by

$$\phi(q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{2}$$

$$\psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{3}$$

$$f(-q) = \sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty. \tag{4}$$

Ramanujan recorded many $q-$continued fractions and some of their explicit values in his second notebook [3] and in his lost notebook [4]. The following beautiful continued fraction identity was recorded by Ramanujan in his second notebook and can be found in [1, p. 11, Entry 11]:

$$\frac{(-a)_\infty(b)_\infty-(a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty+(a)_\infty(-b)_\infty}=\frac{a-b}{1-q}{}_{+}\frac{(a-bq)(aq-b)}{1-q^3}{}_{+}\frac{q(a-bq^2)(aq^2-b)}{1-q^5}{}_{+\cdots} \tag{5}$$

where either q, a, and b are complex numbers with $|q|<1$, or q, a, and b are complex numbers with $a=bq^m$ for some integer m. Several elegant q-continued fractions can be expressed in terms of Ramanujan's theta-functions. The most famous of them is the Rogers-Ramanujan continued fraction $R(q)$ defined by

$$R(q):=\frac{q^{1/5}}{1}{}_{+}\frac{q}{1}{}_{+}\frac{q^2}{1}{}_{+}\frac{q^3}{1}{}_{+\cdots} \tag{6}$$

The continued fraction $R(q)$ has a very beautiful and extensive theory almost all of which was developed by Ramanujan. An account of this can be found in in Chapter 32 of Berndt's book [2].

Now, setting $a=q$ and $b=0$ in (5), we obtain

$$\frac{(-q;q)_\infty-(q;q)_\infty}{(-q;q)_\infty+(q;q)_\infty}=\frac{q}{1-q}{}_{+}\frac{q^3}{1-q^3}{}_{+}\frac{q^5}{1-q^5}{}_{+\cdots} \tag{7}$$

In this paper, we study the Ramanujan's continued fraction $L(q)$ defined by

$$L(q):=\frac{q}{1-q}{}_{+}\frac{q^3}{1-q^3}{}_{+}\frac{q^5}{1-q^5}{}_{+\cdots} \tag{8}$$

which is the right hand side of the equality (7).

In section 2, we prove some modular relations connecting $L(q)$ and $L(q^n)$. In section 3, we give some explicit evaluations of $L(q)$.

We end this introduction by recording some theta-function identities from [1, p. 40, Entry 25]:

$$\phi(q)+\phi(-q)=2\phi(q^4), \tag{9}$$

$$\phi(q)\phi(-q)=\phi(-q^2), \tag{10}$$

$$\phi^2(q)+\phi^2(-q)=2\phi^2(q^2), \tag{11}$$

2. Modular relations of $L(q)$

Theorem 2.1. We have

$$L(q) := \frac{1-\phi(-q)}{1+\phi(-q)}.$$

Proof. Dividing numerator and denominator on left hand side of (7) by $q;q)_\infty$, we obtain

$$L(q) := \left(\frac{(-q;q)_\infty}{(q;q)_\infty} - 1\right) / \left(\frac{(-q;q)_\infty}{(q;q)_\infty} + 1\right) \tag{12}$$

From (1), we have

$$(-q;q)_\infty = (q^2;q^2)_\infty/(q;q)_\infty. \tag{13}$$

Employing (13) in (12) and using (4), we find that

$$L(q) := \left(\frac{f(-q^2)}{f^2(-q)} - 1\right) / \left(\frac{f(-q^2)}{f^2(-q)} + 1\right) \tag{14}$$

Noting $\phi(-q) = \big(f^2(-q)/f(-q^2)\big)$ from [1, p. 39, Entry 24], employing in (14) and simplifying, we complete the proof. ■

Corollary 2.2. We have

$$\frac{1-L(q)}{1+L(q)} = \phi(-q).$$

Proof. Follows easily from Theorem 2.1. ■

From Corollary 2.2, it is obvious that if we have an identity connecting $\phi(-q)$ and $\phi(\pm q^n)$ then we can always find a modular relation between the continued fraction $L(q)$ and the continued fractions $L(\pm q^n)$. We give some examples in the theorem below.

Theorem 2.3. Let $u = L(q), v = L(-q), w = L(q^2), w = L(q^4)$, $r = L(-q^2)$ and $s = L(-q^4)$, then

$$\begin{aligned}
&(i)\ 2s - u + su - v + sv - 2uv = 0,\\
&(ii)\ u + v - 2w - 2uvw + uw^2 + vw^2 = 0,\\
&(iii)\ 2r - u + 2ru - r^2u + 2ru^2 - v + 2rv - r^2v - 4uv - 4r^2uv - u^2v\\
&\qquad + 2ru^2v - r^2u^2v + 2rv^2\\
&\qquad - uv^2 + 2ruv^2 - r^2uv^2 + 2ru^2v^2 = 0,\\
&(iv)\ r - u - r^2u + u^2 + 3ru^2 + r^2u^2 + 2rv - 4uv - 4ruv - 4r^2uv\\
&\qquad + 2ru^2v + v^2 + 3rv^2 + r^2v^2\\
&\qquad - uv^2 - r^2uv^2 + ru^2v^2 = 0.
\end{aligned}$$

Proof. To prove (i), (ii), and (iii), we employ Corollary 2.2 in (9), (10), and (11) respectively. To prove (iv), eliminating $\phi(-q)$ between (9) and (11), we arrive at

$$\phi^2(q)+2\phi^2(q^4)-2\phi(q)\phi(q^4)-\phi^2(q^2)=0. \tag{15}$$

Employing Corollary 2.2 in (15), we complete the proof. ∎

3. Explicit Evaluations of $L(q)$

Theorem 3.1. We have

$$(i)\ \ L(e^{-n\pi})=\frac{1-\phi(-e^{-n\pi})}{1+\phi(-e^{-n\pi})}\ \ \textit{and}\ \ (ii)\ L(-e^{-n\pi})=\frac{1-\phi(e^{-n\pi})}{1+\phi(e^{-n\pi})}.$$

Proof. We set $q:=e^{-n\pi}$ and $q:=-e^{-n\pi}$ in Theorem 2.1 to prove (i) and (ii), respectively. ∎

In his first notebook, Ramanujan [3, Vol. I, p. 248] recorded many elementary values of $\phi(q)$. Particularly, he recorded $\phi(e^{-n\pi})$ and $\phi(-e^{-n\pi})$ for n=1, 2, 4, 8, 1/2, and 1/4. Ramanujan also recorded non-elementary values of $\phi(e^{-n\pi})$ for n= 3, 5, 7, 9, and 45. Proofs of these can be found in [2]. Yi [5] also evaluated $\phi(e^{-n\pi})$ for n=1, 2, 3, 4, 5, and 6 and $\phi(-e^{-n\pi})$ for n=1, 2, 4, 6, 8, 10, and 12. Noting from [2, p. 325, Entry 1(i) & (ii)], we have

$$\phi(e^{-\pi})=a \quad\text{and}\quad \phi(-e^{-\pi})=2^{-1/4}a, \tag{16}$$

where

$$a=\pi^{1/4}/\Gamma(3/4). \tag{17}$$

Employing (16) in Theorem 3.1, we obtain

$$L(-e^{-\pi})=\frac{1-a}{1+a}\quad\text{and}\quad L(e^{-\pi})=\frac{2^{1/4}-a}{2^{1/4}+a}.$$

References

[1] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[2] B. C. Berndt, *Ramanujan's Notebooks, Part V*, Springer-Verlag, New York, 1998.

[3] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[4] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[5] J. Yi, Theta-function identities and the explicit formulas for theta-function and their applications, *J. Math. Anal. Appl.*, 292:381–400, 2004.