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Oscillation, Nonoscillation and Growth of Solutions of Generalized Second Order Nonlinear α-difference Equation

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Abstract

In this paper, the authors discuss the oscillation, nonoscillation and growth of the solutions of the generalized nonlinear α-difference equation

$$\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}u(k)) + f(k)F(k, u(k), \Delta_{\alpha(\ell)}u(k)) = g(k, u(k), \Delta_{\alpha(\ell)}u(k)), \alpha > 1, \tag{1}$$

$k \in [a, \infty)$, where the functions p, f, F and g are defined in their domain of definition and ℓ is a positive real. Further, $p(k) > 0$ for all $k \in [a, \infty)$ for some $a \in [0, \infty)$ and for all $j = k - a - [\frac{k-a}{\ell}]\ell, R_{a+j,k} \to \infty$, where $R_{t+j,k} = \sum_{r=0}^{\frac{k-\ell-t-j}{\ell}} \frac{1}{p(t+j+r\ell)}$, $t \in [a, \infty)$ and $k \in \mathbb{N}_\ell(t + j + \ell)$.

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1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k)$, $k \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Eventhough many authors ([1], [16]-[18]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k),\ k \in \mathbb{R},\ \ell \in \mathbb{R} - \{0\}, \tag{2}$$

and no significant progress has taken place on this line. But recently, E. Thandapani, M.M.S. Manuel, G.B.A.Xavier [9] considered the definition of Δ as given in (2) and developed the theory of difference equations in a different direction. For convenience, the operator Δ defined by (2) is labelled as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory (see [9], [12]-[15]) were obtained. By extending the study related to the sequences of complex numbers and ℓ to be real, some new qualitative properties of the solutions like rotatory, expanding, shrinking, spiral and weblike of difference equations involving Δ_ℓ were obtained. The results obtained using Δ_ℓ can be found in ([9]-[15]). Jerzy Popenda and B.Szmanda ([6],[7]) defined Δ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$ and based on this definition they have studied the qualitative properties of the difference equation and no one else has handled this operator. Here, the generalized definition of the operator is used as

$$\Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k). \tag{3}$$

and by defining its inverse, several interesting results on number theory were obtained.

John R. Graef [8] worked on Oscillation, nonoscillation, and growth of solutions of nonlinear functional differential equations of arbitrary order and Blazej Szmanda [3] discussed oscillation, nonoscillation and growth of solution of (1) with $\ell = 1, \alpha = 1$ and $k \in \mathbb{N}(a)$ which is the discrete analogue of [8]. The case of any real ℓ and $\alpha = 1$, in (1) were analysed in detail by M.M.S.Manuel, D.S. Dilip, etal [13]. In this paper the theory is extended from Δ_ℓ to $\Delta_{\alpha(\ell)}$ for all real $k \in [a, \infty)$ and the oscillation, nonoscillation and growth of solutions of the generalized nonlinear α-difference equation (1) is discussed. Our results generalize those found in ([4],[5]). Suitable examples are provided to illustrate our main results.

Throughout this paper, we make use of the following notations;

(i) $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $\mathbb{N}(a) = \{a, a+1, a+2, \dots\}$,

(ii) $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \dots\}$, $j = k - a - \left[\dfrac{k-a}{\ell}\right]\ell$, for $k \in [a, \infty)$,

(iii) $\mathbb{C}$ is set of all complex numbers, $\mathbb{R}$ is the set of all real numbers and

(iv) $[X]$ denotes the integer part of X,

(v) $\Omega^\alpha_{a+j,k}(p, \phi, M_2, f) = \displaystyle\sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{\alpha^{\left\lceil \frac{k}{\ell} \right\rceil}}{p(a+j+r\ell)} \sum_{i=0}^{r-1} \frac{(\phi(a+j+i\ell) - M_2 f(a+j+i\ell))}{\alpha^{\left\lceil \frac{a+j+2\ell+i\ell}{\ell} \right\rceil}}$.

2. Preliminaries

In this section we present some basic definitions and some results already obtained which is useful for further discussion.

Definition 2.1. Let $u(k)$, $k \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the generalized α-difference operator $\Delta_{\alpha(\ell)}$ on $u(k)$ is defined as

$$\Delta_{\alpha(\ell)} u(k) = u(k + \ell) - \alpha u(k). \tag{4}$$

When $\alpha = 1$, the generalized α-difference operator $\Delta_{\alpha(\ell)}$ becomes the generalized difference operator Δ_ℓ. When $\alpha = 1$ and $\ell = 1$, $\Delta_{\alpha(\ell)}$ assumes the role of the usual difference operator Δ.

Definition 2.2. [9] Let $u(k), k \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as follows.

$$\text{If } \Delta_\ell v(k) = u(k), then v(k) = \Delta_\ell^{-1} u(k) + c_j, \tag{5}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - \left[\frac{k}{\ell}\right]\ell$. In general, $\Delta_\ell^{-n} u(k) = \Delta_\ell^{-1}(\Delta_\ell^{-(n-1)} u(k))$ for $n \in \mathbb{N}(2)$.

Definition 2.3. The inverse of the Generalized α-difference operator denoted by $\Delta_{\alpha(\ell)}^{-1}$ on $u(k)$ is defined as, if $\Delta_{\alpha(\ell)} v(k) = u(k)$, then

$$\Delta_{\alpha(\ell)}^{-1} u(k) = v(k) - \alpha^{\left[\frac{k}{\ell}\right]} c_j \tag{6}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - a - [\frac{k-a}{\ell}]\ell$.

Lemma 2.4. [9]If the real valued function $u(k)$ is defined for all $k \in [a, \infty)$, then

$$\Delta_\ell^{-1} u(k) = \sum_{r=1}^{\left[\frac{k-a}{\ell}\right]} u(k - r\ell) + c_j, \tag{7}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - a - [\frac{k-a}{\ell}]\ell$.

Corollary 2.5. If $\Delta_\ell v(k) = u(k)$ for $k \in [k_2, \infty)$ and $j = k - k_2 - [\frac{k-k_2}{\ell}]\ell$, then

$$v(k) - v(k_2 + j) = \sum_{r=0}^{\left[\frac{k-k_2-j-\ell}{\ell}\right]} u(k_2 + j + r\ell).$$

Proof. The proof follows by Definition 2.2, Lemma 2.4 and $c_j = v(k_2 + j)$. ■

Definition 2.6. The solution $u(k)$ of (1) is called oscillatory if for any $k_1 \in [a, \infty)$ there exists a $k_2 \in \mathbb{N}_\ell(k_1)$ such that $u(k_2)u(k_2+\ell) \leq 0$. The difference equation itself is called oscillatory if all its solutions are oscillatory. If the solution $u(k)$ is not oscillatory then it is said to be nonoscillatory (i.e. $u(k)u(k+\ell) > 0$ for all $k \in [k_1, \infty)$).

Lemma 2.7. The relation between Δ_ℓ and $\Delta_{\alpha(\ell)}$ is given by

$$\alpha^{\lceil \frac{k+\ell}{\ell} \rceil} \Delta_\ell \Big(\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} \Big) = \Delta_{\alpha(\ell)} u(k).$$

Lemma 2.8. [2] (Discrete Gronwall's Inequality) Let $u(k)$ and $v(k)$ be nonnegative functions defined on $[0, \infty)$ and $c \geq 0$ be a constant. If

$$u(k) \leq c + \sum_{r=0}^{k-1} v(r)u(r) \text{ for } k \in [0, \infty), \tag{8}$$

then

$$u(k) \leq c \exp\left(\sum_{r=0}^{t-1} v(r) \right) \text{ for } k \in [0, \infty). \tag{9}$$

Lemma 2.9. (Discrete Generalized Gronwall's Inequality) Let $u(k)$ and $v(k)$ be nonnegative functions defined on $[0, \infty)$ and $c \geq 0$ be a constant. If

$$u(k) \leq c + \sum_{r=0}^{\frac{k-a-\ell-j}{\ell}} v(a+j+r\ell)u(a+j+r\ell) \text{ for } k \in [a, \infty), \tag{10}$$

then

$$u(k) \leq c \exp\left(\sum_{r=0}^{\frac{k-a-\ell-j}{\ell}} v(a+j+r\ell) \right) \text{ for } k \in [a, \infty). \tag{11}$$

Proof. The proof follows from Lemma 2.8 by taking $u(r)$ as $u(a+j+r\ell)$ and $v(r)$ as $v(a+j+r\ell), r \in \mathbb{N},\ j = k - a - [\frac{k-a}{\ell}]\ell$. ■

In order to prove the main results one or more of the following conditions have been used.

(c_1) $f(k) \geq 0$ for all $k \in [a, \infty)$,

(c_2) there exists a constant M_1 such that $F(k, u, v) \geq M_1$,

(c_3) there exists a constant M_2 such that $F(k,u,v) \leq M_2$

(c_4) there exists a constant $M > 0$ such that $|F(k,u,v)| \leq M$,

(c_5) there exists a function $\phi(k)$ such that $g(k,u,v) \geq \phi(k)$,

(c_6) there exists a function $\psi(k)$ such that $g(k,u,v) \leq \psi(k)$,

(c_7) $F(k,u,v)$ is bounded from above if u is bounded,

(c_8) $F(k,u,v)$ is bounded from below if u is bounded,

(c_9) $uF(k,u,v) \geq 0$,

(c_{10}) $uF(k,u,v) \leq 0$,

(c_{11}) there exist functions $m(k)$ and $n(k)$ such that $m(k) \leq F(k,u,v) \leq n(k)$,

(c_{12}) there exists a nonnegative real sequence $B(k)$ such that $|g(k,u,v)| \leq B(k)$,

(c_{13}) there exists a nonnegative real sequence $m(k)$ such that $|F(k,u,v)| \leq m(k)|u|$.

3. Main Results

In this section we discuss the oscillation, nonoscillation and growth of solutions of equation (1).

3.1. Nonoscillation Results

Theorem 3.1. Let $k \in \mathbb{N}_\ell(a+j+\ell)$ and $j \in [0,\ell)$. Suppose that conditions (c_1),(c_3) and (c_5) hold and for every constant $C > 0$,

$$\liminf_{k\to\infty} \left(\Omega^{\alpha}_{a+j,k}(p,\phi,M_2,f) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C R_{a+j,k}\right) > 0, \text{ for all } 0 \leq j < \ell. \tag{12}$$

Then, all solutions of (1) are eventually positive.

Proof. Let $u(k)$ be a solution of (1). Applying conditions (c_1), (c_3) and (c_5), we obtain

$$\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}u(k)) \geq \phi(k) - M_2 f(k),\ k \in [a,\infty).$$

Therefore, it follows from Corollary 2.5 and Lemma 2.7 that, for all $0 \leq j < \ell$, $k \in \mathbb{N}_\ell(a+\ell+j)$,

$$\begin{aligned}\Delta_{\alpha(\ell)}u(k) \geq{}& \frac{\alpha^{\left\lceil \frac{k}{\ell} \right\rceil} p(a+j)}{\alpha^{\left\lceil \frac{a+j}{\ell} \right\rceil} p(k)} \Delta_{\alpha(\ell)}u(a+j) \\ &+ \frac{\alpha^{\left\lceil \frac{k}{\ell} \right\rceil}}{p(k)} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{(\phi(a+j+r\ell) - M_2 f(a+j+r\ell))}{\alpha^{\left\lceil \frac{a+j+\ell+r\ell}{\ell} \right\rceil}}.\end{aligned}$$

Again applying Corollary 2.5 and Lemma 2.7, we obtain

$$u(k) \geq \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} \left(\frac{u(a+j)}{\alpha^{\left\lceil \frac{a+j}{\ell} \right\rceil}} + \frac{p(a+j)\Delta_{\alpha(\ell)} u(a+j)}{\alpha^{\left\lceil \frac{a+j+\ell}{\ell} \right\rceil}} R_{a+j,k} \right) + \Omega^{\alpha}_{a+j,k}(p, \phi, M_2, f).$$

Now, in view of the conditions on $p(k)$, there exist a constant $C > 0$ and $k_1 \in [a, \infty)$ such that

$$-CR_{a+j,k} \leq \frac{u(a+j)}{\alpha^{\left\lceil \frac{a+j}{\ell} \right\rceil}} + \frac{p(a+j)}{\alpha^{\left\lceil \frac{a+j+\ell}{\ell} \right\rceil}} \Delta_{\alpha(\ell)} u(a+j) R_{a+j,k} \leq CR_{a+j,k}, \tag{13}$$

$k \in [k_1, \infty)$ and $j = k - a - \left[\frac{k-a}{\ell} \right] \ell$. Hence, from (12) we have

$$\lim_{k\to\infty} \inf u(k) \geq \lim_{k\to\infty} \inf \left(\Omega^{\alpha}_{a+j,k}(p, \phi, M_2, f) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} CR_{a+j,k} \right) > 0.$$

Thus, $u(k)$ is eventually positive, which completes the proof. ■

Example 3.2. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}(k\Delta_{\alpha(\ell)} u(k)) = \frac{(k+\ell)^2 - \alpha k(k+2\ell)}{(k+2\ell)^{(2)}_{\ell}} - \alpha(1-\alpha)$$

and for

$$f = 0, \phi = \frac{(k+\ell)^2 - \alpha(k(k+2\ell))}{(k+2\ell)^{(2)}_{\ell}} - \alpha(1-\alpha),$$

the conditions of Theorem 3.1 hold and hence all the solutions are eventually positive. Infact $u(k) = \frac{1}{k}$ is one solution.

Remark 3.3. If we replace (12) in Theorem 3.1 by the stronger condition

$$\sum_{r=0}^{\infty} \frac{(\phi(a+j+r\ell) - M_2 f(a+j+r\ell))}{\alpha^{\left\lceil \frac{a+j+\ell+r\ell}{\ell} \right\rceil}} = \infty,$$

then, every solution $u(k)$ of (1) satisfies $\lim_{k\to\infty} u(k) = \infty$ monotonically. Indeed, then from (1), we obtain

$$p(k)\Delta_{\alpha(\ell)} u(k) \geq \frac{\alpha^{\left\lceil \frac{k}{\ell} \right\rceil} p(a+j)}{\alpha^{\left\lceil \frac{a+j}{\ell} \right\rceil}} \Delta_{\alpha(\ell)} u(a+j) + \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{(\phi(a+j+r\ell) - M_2 f(a+j+r\ell))}{\alpha^{\left\lceil \frac{a+j+\ell+r\ell}{\ell} \right\rceil}} \to \infty$$

as $k \to \infty$. Thus, there exists a $k_1 \in [a, \infty)$ such that $\Delta_{\alpha(\ell)} u(k) \geq \dfrac{1}{p(k)}$ for all $k \in [k_1, \infty)$, from which the conclusion follows.

The proof of the following results are similar to that of Theorem 3.1 and therefore are omitted.

Theorem 3.4. Suppose that conditions (c_1), (c_2) and (c_6) hold and for every constant $C > 0$

$$\lim_{k\to\infty} \sup \left(\Omega^{\alpha}_{a+j,k}(p, \psi, M_1, f) + \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C R_{a+j,k}\right) < 0,$$
$$k \in [a, \infty),\ j = k - a - \left\lceil \frac{k-a}{\ell} \right\rceil \ell. \tag{14}$$

Then, all solutions of (1) are eventually negative.

Example 3.5. For the generalized $\alpha-$difference equation $\Delta_{\alpha(\ell)}(k\Delta_{\alpha(\ell)} u(k)) + \alpha^2 k^3 = -(k+\ell)((k+2\ell)^2 - \alpha(k+\ell)(2k+\ell))$ and for $M_1 = \alpha^2, f = 1, \psi = -(k+\ell)((k+2\ell)^2 - \alpha(k+\ell)(2k+\ell))$, the conditions of Theorem 3.4 hold and therefore all the solutions are eventually negative. Infact $u(k) = -k^2$ is one solution.

Remark 3.6. In Theorem 3.4 if we replace (14) by the condition

$$\sum_{r=0}^{\infty} \frac{(\psi(a+j+r\ell) - M_2 f(a+j+r\ell))}{\alpha^{\left\lceil \frac{a+j+\ell+r\ell}{\ell} \right\rceil}} = -\infty,$$

then, every solution of (1) satisfies $\lim_{k\to\infty} u(k) = -\infty$ monotonically.

Theorem 3.7. Suppose that conditions (c_1),(c_5) and (c_7) hold and for every constant $C_1, C_2 > 0$

$$\lim_{k\to\infty} \inf \left(\Omega^{\alpha}_{a+j,k}(p, \phi, C_1, f) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C_2 R_{a+j,k}\right) > 0. \tag{15}$$

Then, all bounded solutions of (1) are eventually positive.

Example 3.8. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\left(\frac{1}{k}\Delta_{\alpha(\ell)} u(k)\right) = \frac{(1-2\alpha)(k(1-2\alpha) - 2\alpha\ell)}{2^{\left\lceil \frac{k+2\ell}{\ell} \right\rceil}(k+\ell)^{(2)}_{\ell}}$$

and for

$$f = 0, \phi = \frac{(1-2\alpha)(k(1-2\alpha) - 2\alpha\ell)}{2^{\left\lceil \frac{k+2\ell}{\ell} \right\rceil}(k+\ell)^{(2)}_{\ell}},$$

the conditions of Theorem 3.7 hold and consequently all the solutions are eventually positive. One such solution is $u(k) = \frac{1}{2^{\left\lceil \frac{k}{\ell} \right\rceil}}$.

Theorem 3.9. Suppose that conditions (c_1), (c_6) and (c_8) hold and for every constants $C_1, C_2 > 0$

$$\lim_{k\to\infty} \sup \left(\Omega^{\alpha}_{a+j,k}(p, \psi, C_1, f) + \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C_2 R_{a+j,k}\right) < 0. \tag{16}$$

Then, all bounded solutions of (1) are eventually negative.

Example 3.10. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\left(\frac{1}{k}\Delta_{\alpha(\ell)}u(k)\right) = \frac{(2\alpha-1)(k(1-2\alpha)-2\alpha\ell)}{2^{\left\lceil \frac{k+2\ell}{\ell} \right\rceil}(k+\ell)^{(2)}_{\ell}}$$

and for

$$f = 0, \phi = \frac{(2\alpha-1)(k(1-2\alpha)-2\alpha\ell)}{2^{\left\lceil \frac{k+2\ell}{\ell} \right\rceil}(k+\ell)^{(2)}_{\ell}},$$

the conditions of Theorem 3.9 hold and ultimately all the solutions are eventually negative. $u(k) = \frac{-1}{2^{\left\lceil \frac{k}{\ell} \right\rceil}}$ is one solution.

Theorem 3.11. Suppose that conditions (c_4) and (c_5) hold and for every constant $C > 0$

$$\lim_{k\to\infty} \inf \left(\Omega^{\alpha}_{a+j,k}(p, \phi, M, |f|) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C R_{a+j,k}\right) > 0. \tag{17}$$

Then, all solutions of (1) are eventually positive.

Example 3.8 illustrates Theorem 3.11.

Theorem 3.12. Suppose that conditions (c_4) and (c_6) hold and for every constant $C > 0$,

$$\lim_{k\to\infty} \sup \left(\Omega^{\alpha}_{a+j,k}(p, \psi, M, |f|) + \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C R_{a+j,k}\right) < 0. \tag{18}$$

Then, all solutions of (1) are eventually negative.

Example 3.13. For the generalized $\alpha-$difference equation $\Delta_{\alpha(\ell)}(k\Delta_{\alpha(\ell)}u(k)) = (\alpha-2)2^{\left\lceil \frac{k}{\ell} \right\rceil}(k(2-\alpha)+2\ell)$ and for $f = 0, \psi = (\alpha-2)2^{\left\lceil \frac{k}{\ell} \right\rceil}(k(2-\alpha)+2\ell)$, the conditions of Theorem 3.12 holds and all the solutions are eventually negative. one such solution is $u(k) = -2^{\left\lceil \frac{k}{\ell} \right\rceil}$.

Remark 3.14. Replacing (17) and (18) by

$$\sum_{r=0}^{\infty} \frac{(\phi(a+j+r\ell) - M_2 f(a+j+r\ell))}{\alpha^{\left\lceil \frac{a+j+\ell+r\ell}{\ell} \right\rceil}} = \infty$$

and

$$\sum_{r=0}^{\infty} \frac{(\psi(a+j+r\ell) - M_2 f(a+j+r\ell))}{\alpha^{\left\lceil \frac{a+j+\ell+r\ell}{\ell} \right\rceil}} = -\infty$$

respectively in Theorem 3.12 we obtain an analogous results to those in Remarks 3.3 and 3.6.

Theorem 3.15. Suppose that conditions (c_5),(c_7) and (c_8) hold and for every constants $C_1, C_2 > 0$

$$\lim_{k\to\infty} \inf \left(\Omega^{\alpha}_{a+j,k}(p, \phi, C_1, |f|) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C_2 R_{a+j,k}\right) > 0. \tag{19}$$

Then, all bounded solutions of (1) are eventually positive.

Example 3.8 illustrates Theorem 3.15.

Theorem 3.16. Suppose that conditions (c_6) - (c_8) hold and for every constant $C_1, C_2 > 0$

$$\lim_{k\to\infty} \sup \left(\Omega^{\alpha}_{a+j,k}(p, \psi, C_1, |f|) + \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C_2 R_{a+j,k}\right) < 0. \tag{20}$$

Then, all bounded solutions of (1) are eventually negative.

Example 3.17. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}(k\Delta_{\alpha(\ell)}u(k)) - \frac{\alpha(2k+\ell)}{(k+\ell)^2} = -\left[\frac{k+\ell}{(k+2\ell)^2} + \frac{\alpha^2}{k}\right]$$

and for

$$C_1 = 1, f = -\alpha, \psi = -\left[\frac{k+\ell}{(k+2\ell)^2} + \frac{\alpha^2}{k}\right],$$

the conditions of Theorem 3.16 hold and therefore all the solutions are eventually negative. $u(k) = \frac{-1}{k^2}$ is a solution of the given equation.

3.2. Oscillation Results

In this section, we provide sufficient conditions for the oscillation of solutions of equation (1).

Theorem 3.18. Suppose that $f(k) \equiv 1$ and conditions (c_5), (c_6) and (c_{11}) hold. Further, let for every constant $C > 0$ and for all large $s \in [a, \infty)$

$$\lim_{k\to\infty} \inf \left(\Omega^{\alpha}_{s+j,k}(p, \psi, 1, m) + \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C R_{s+j,k}\right) < 0. \tag{21}$$

and

$$\lim_{k\to\infty} \sup \left(\Omega^{\alpha}_{s+j,k}(p, \phi, 1, n) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C R_{s+j,k}\right) > 0. \tag{22}$$

Then, the difference equation (1) is oscillatory.

Proof. Let $u(k)$ be a nonoscillatory solution of (1), say $u(k) > 0$ for all $s \le k \in [a, \infty)$. Then, from (1) we have

$$\phi(k) - n(k) \le \Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}u(k)) \le \psi(k) - m(k),\ k \in [s, \infty).$$

Now following as in Theorem 3.1

$$\Omega^{\alpha}_{s+j,k}(p, \phi, 1, n) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C R_{s+j,k} \le u(k) \le \Omega^{\alpha}_{s+j,k}(p, \psi, 1, m) + \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C R_{s+j,k}.$$

(21) then yields a contradiction to the assumption that $u(k) > 0$ for all $k \in [s, \infty)$. A similar proof holds if $u(k) < 0$ for all $k \in [s, \infty)$. This completes the proof of the theorem. ■

Example 3.19. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}(k\Delta_{\alpha(\ell)}u(k)) = (2+\alpha)((2+\alpha)k + 2\ell)(-2)^{\left\lceil \frac{k}{\ell} \right\rceil},$$

and for

$$m(k) = n(k) = 0, \psi = (2+\alpha)((2+\alpha)k + 4\ell)(-2)^{\left\lceil \frac{k}{\ell} \right\rceil}, \phi = (2+\alpha)^2 k(-2)^{\left\lceil \frac{k}{\ell} \right\rceil},$$

the conditions of Theorem 3.18 hold and hence the generalized difference equation is oscillatory. Infact $u(k) = (-2)^{\left\lceil \frac{k}{\ell} \right\rceil}$ is a solution.

Theorem 3.20. Suppose that $f(k) \equiv 1$ and conditions (c_5), (c_6) and (c_9) hold. Further, let for every constant $C > 0$ and for all large $s \in [a, \infty)$

$$\lim_{k\to\infty} \inf \left(\Omega^{\alpha}_{s+j,k}(p, \psi, 0, 0) + \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C R_{s+j,k}\right) < 0. \tag{23}$$

and

$$\lim_{k\to\infty}\sup\left(\Omega^{\alpha}_{s+j,k}(p,\phi,0,0)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}CR_{s,k}\right)>0. \tag{24}$$

Then, the difference equation (1) is oscillatory.

Proof. Let $u(k)$ be a nonoscillatory solution of (1), say $u(k)\geq 0$ for $s\leq k\in[a,\infty)$. From (1) we have for $k\geq s$

$$\Delta_{\alpha(\ell)}(p(k)\Delta_{\alpha(\ell)}u(k))=-F(k,u(k),\Delta_{\alpha(\ell)}u(k))+g(k,u(k),\Delta_{\alpha(\ell)}u(k))\leq\psi(k).$$

Proceeding as before, we observe that there exist $C>0$ and $k_1\in[s,\infty)$ such that an estimate of type (13) holds for $k\geq k_1$ and

$$u(k)\leq\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}CR_{s+j,k}+\Omega^{\alpha}_{s+j,k}(p,\psi,0,0).$$

Then (23) gives a contradiction to the assumption that $u(k)\geq 0$ for $k\geq s$. The proof in case $u(k)\leq 0$ for $k\geq s$ is similar, which completes the proof. ■

Example 3.21. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}(k\Delta_{\alpha(\ell)}u(k))=(1+\alpha)(k(1+\alpha)+\ell)(-1)^{\left\lceil\frac{k}{\ell}\right\rceil}$$

and for

$$\psi=(1+\alpha)(k(1+\alpha)+2\ell)(-1)^{\left\lceil\frac{k}{\ell}\right\rceil},\phi=k(1+\alpha)^2(-1)^{\left\lceil\frac{k}{\ell}\right\rceil},$$

the conditions of Theorem 3.20 hold and the generalized difference equation is oscillatory. $u(k)=(-1)^{\left\lceil\frac{k}{\ell}\right\rceil}$ is one solution.

Theorem 3.22. Suppose that $f(k)\equiv 1$ and conditions (c_5), (c_6) and (c_{10}) hold. Further, let for every constant $C>0$ and for all large $s\in[a,\infty)$

$$\lim_{k\to\infty}\inf\left(\Omega^{\alpha}_{s+j,k}(p,\psi,0,0)+\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}CR_{s+j,k}\right)=-\infty. \tag{25}$$

and

$$\lim_{k\to\infty}\sup\left(\Omega^{\alpha}_{s+j,k}(p,\phi,0,0)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}CR_{s+j,k}\right)=\infty. \tag{26}$$

Then, all the bounded solutions of the difference equation (1) are oscillatory.

Proof. Let $u(k)$ be a nonoscillatory solution of (1), say $u(k)\geq 0$ for $k\in[s,\infty)(s\geq 0)$ and assume that $u(k)$ is bounded. Arguing as in the proof of the previous theorems we obtain the inequality

$$u(k)\geq\Omega^{\alpha}_{s+j,k}(p,\psi,0,0)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}CR_{s+j,k}\text{ for }k\in[k_1,\infty)(k_1\geq s).$$

Then (26) gives a contradiction to the boundedness of $u(k)$. A similar argument treats the case of an eventually non positive solution. This completes the proof. ■

Example 3.21 illustrates Theorem 3.22.

3.3. Growth of Solutions

In this section we provide sufficient conditions for the asymptotic nature of the solutions of equation (1).

Theorem 3.23. Suppose that conditions (c_{12}) and (c_{13}) hold and for $j \in (0, \ell)$

$$\sum_{r=0}^{\infty} B(a+j+r\ell) < \infty, \ \sum_{r=0}^{\infty} P(a+j+r\ell) R_{a+j,k} < \infty. \tag{27}$$

Then, every solution $u(k)$ of (1) satisfies $|u(k)| = o(R_{a+j,k})$ as $n \to \infty$.

Proof. Let $u(k)$ be a solution of (1). From (1), by successive summations, we obtain

$$\begin{aligned} u(k) \geq \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} & \left(\frac{u(a+j)}{\alpha^{\left\lceil \frac{a+j}{\ell} \right\rceil}} + \frac{p(a+j)\Delta_{\alpha(\ell)} u(a+j)}{\alpha^{\left\lceil \frac{a+j}{\ell} \right\rceil}} R_{a+j,k} \right) \\ & + \Omega^{\alpha}_{a+j,k}(p, g(a+j+i\ell, u(a+j+i\ell), \Delta_{\alpha(\ell)} u(a+j+i\ell)), \\ & F(a+j+i\ell, u(a+j+i\ell), \Delta_{\alpha(\ell)} u(a+j+i\ell)), 1) \end{aligned}$$

and so

$$\begin{aligned} |u(k)| \leq \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} C_1 R_{a+j,k} & + \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{\alpha^{\left\lceil \frac{k}{\ell} \right\rceil}}{p(a+j+r\ell)} \sum_{i=0}^{r-1} \frac{m(a+j+i\ell)|u(a+j+i\ell)|}{\alpha^{\left\lceil \frac{a+j+2\ell+i\ell}{\ell} \right\rceil}} \\ & + \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{\alpha^{\left\lceil \frac{k}{\ell} \right\rceil}}{p(a+j+r\ell)} \sum_{i=0}^{r-1} \frac{B(a+j+i\ell)}{\alpha^{\left\lceil \frac{a+j+2\ell+i\ell}{\ell} \right\rceil}} \end{aligned}$$

for some constant $C_1 > 0$ and all $k \geq k_1 \in [a, \infty)$. By the first condition in (27), there is a constant $C_2 > 0$ such that

$$\sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{1}{p(a+j+r\ell)} \sum_{i=0}^{r-1} \frac{B(a+j+i\ell)}{\alpha^{\left\lceil \frac{a+j+2\ell+i\ell}{\ell} \right\rceil}} \leq C_2 R_{a+j,k}.$$

Thus for $k \geq k_1$ we obtain

$$|u(k)| \leq C R_{a+j,k} + R_{a+j,k} \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{m(a+j+i\ell)|u(a+j+i\ell)|}{\alpha^{\left\lceil \frac{a+j+2\ell+i\ell}{\ell} \right\rceil}}$$

for some constant $C > 0$, and so

$$|u(k)|/R_{a+j,k} \leq C + \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{m(a+j+i\ell)R_{a+j,k}|u(a+j+i\ell)|}{R_{a+j,k}\alpha^{a+j+2\ell+i\ell}}.$$

Now using the discrete Gronwall's inequality (Lemma 2.9) one has

$$|u(k)|/R_{a+j,k} \leq C \exp \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{m(a+j+i\ell)R_{a+j,k}}{\alpha^{\left\lceil \frac{a+j+2\ell+i\ell}{\ell} \right\rceil}}$$

and the conclusion of the theorem follows from (27). ■

Example 3.24. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}\left(\frac{1}{k}\Delta_{\alpha(\ell)}u(k)\right) = \frac{(1-2\alpha)(k(1-2\alpha)-2\alpha\ell)}{2^{\left\lceil \frac{k+2\ell}{\ell} \right\rceil}(k+\ell)_\ell^{(2)}}$$

and for

$$B = \frac{(1-2\alpha)(k(1-2\alpha)-2\alpha\ell)}{2^{\left\lceil \frac{k+2\ell}{\ell} \right\rceil}(k+\ell)_\ell^{(2)}},$$

the conditions of Theorem 3.23 holds and every solution satisfies $o(R_{a+j,k})$ as $n \to \infty$. Infact $u(k) = \dfrac{1}{2^{\left\lceil \frac{k}{\ell} \right\rceil}}$ is a solution of the difference equation.

Theorem 3.25. If the conditions (C_{12}) and (C_9) hold, then any nonoscillatory solution $u(k)$ of (1) satisfies

$$|u(k)| = o\big(\alpha^{\left\lceil \frac{k}{\ell} \right\rceil} R_{a+j,k} + \Omega^{\alpha}_{a+j,k}(p, B, 0, 0)\big) \text{ as } k \to \infty. \tag{28}$$

Proof. Let $u(k)$ be a nonoscillatory solution of (1), then $u(k) \geq 0$ or $u(k) \leq 0$ for all $k \geq s \in [a, \infty)$. If $u(k) \geq 0$, then from (1) we obtain

$$0 \leq u(k) \leq \alpha^{\left\lceil \frac{k}{\ell} \right\rceil}\left(\frac{u(s+j)}{\alpha^{\left\lceil \frac{s+j}{\ell} \right\rceil}} + \frac{p(s+j)\Delta_{\alpha(\ell)}u(s+j)}{\alpha^{\left\lceil \frac{s+j}{\ell} \right\rceil}}R_{s+j,k}\right) + \Omega^{\alpha}_{s+j,k}(p, B, 0, 0)$$

and, if $u(k) \leq 0$,

$$0 \geq u(k) \geq \alpha^{\left\lceil \frac{k}{\ell} \right\rceil}\left(\frac{u(s+j)}{\alpha^{\left\lceil \frac{s+j}{\ell} \right\rceil}} + \frac{p(s+j)\Delta_{\alpha(\ell)}u(s+j)}{\alpha^{\left\lceil \frac{s+j}{\ell} \right\rceil}}R_{s+j,k}\right) - \Omega^{\alpha}_{s+j,k}(p, B, 0, 0).$$

Hence in either case

$$|u(k)| \leq \alpha^k \left(\frac{|u(s+j)|}{\alpha^{\left\lceil \frac{s+j}{\ell} \right\rceil}} + \frac{p(s+j)|\Delta_{\alpha(\ell)} u(s+j)|}{\alpha^{\left\lceil \frac{s+j}{\ell} \right\rceil}} R_{s+j,k} \right) + \Omega^{\alpha}_{s+j,k}(p, B, 0, 0)$$

so the conclusion of the theorem follows immediately. ■

Example 3.26. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)} \left(\frac{1}{k} \Delta_{\alpha(\ell)} u(k) \right) = \frac{(2\alpha - 1)(k(1 - 2\alpha) - 2\alpha\ell)}{2^{\left\lceil \frac{k+2\ell}{\ell} \right\rceil} (k + \ell)^{(2)}_{\ell}}$$

and for

$$B(k) = \frac{(2\alpha - 1)(k(1 - 2\alpha) - 2\alpha\ell)}{2^{\left\lceil \frac{k+2\ell}{\ell} \right\rceil} (k + \ell)^{(2)}_{\ell}},$$

the conditions of Theorem 3.25 hold and hence any nonoscillatory solution satisfies (28). One such solution of the difference equation is $u(k) = \frac{-1}{2^{\left\lceil \frac{k}{\ell} \right\rceil}}$.

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