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Distributive Convex ℓ-Submodule

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Abstract

In this paper generalization of distributive ℓ-ideal for convex ℓ-submodule called distributive convex ℓ-submodule is introduced. Many important properties of distributive ℓ-ideal which are valid for distributive convex ℓ-submodule are established. It's relation with congruence class are also established.

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1. Preliminaries

The definitions and results for ℓ-module, properties of ℓ-module and ℓ-ideal in ℓ-module, Distributive ℓ-ideal in ℓ-module are given in [7], [8], [9] and [10]. From the results obtained by the above papers we give the generalization for convex ℓ-submodule, some of the definitions which are used in the previous papers are given below.

Definition 1.1. A non-empty set M is said to be a ℓ-module over a ring R, if it is equipped with the binary operation $+$, s.m and binary relation $\leq$ defined on it and satisfy the following condition

(i) M is a module over a ring R.

(ii) $(M, \leq)$ is a lattice.

(iii) $x \leq y \Rightarrow a + x \leq a + y$, for all $a, x, y \in M$

(iv) $x \leq y \Rightarrow \alpha x \leq \alpha y$, for all $x, y \in M$ and $\alpha \in R$, with $\alpha \geq 0$.

Definition 1.2. A non-empty set M is said to be a ℓ-module over a ring R, if it is equipped with binary operation $+$, s.m, $\vee$ and $\wedge$ defined on it and satisfy the following conditions

(i) M is a module over a ring R.

(ii) $(M, \vee, \wedge)$ is a lattice.

(iii) $a + (x \vee y) = (a + x) \vee (a + y)$
$a + (x \wedge y) = (a + x) \wedge (a + y)$,for all $a, x, y \in M$

(iv) $\alpha(x \vee y) = \alpha x \vee \alpha y$
$\alpha(x \wedge y) = \alpha x \wedge \alpha y$
for all $x, y \in M$ and $\alpha \in R$ with $\alpha \geq 0$

Properties of ℓ-module

(i) $[(a - b) \vee 0] + b = a \vee b$,for all a, b in M

(ii) $a \leq b \Rightarrow a - c \leq b - c$ and $c - b \leq c - a$ for all a, b, c in M

(iii) $(a \vee b) - c = (a - c) \vee (b - c)$, for all a, b, c in M

(iv) $a - (b \vee c) = (a - b) \wedge (a - c)$, for all a, b, c in M

(v) $a - (b \wedge c) = (a - b) \vee (a - c)$, for all a, b, c in M

(vi) $(b \wedge c) - a = (b - a) \wedge (c - a)$, for all a, b, c in M

(vii) $a \leq b \Rightarrow (a - b) + b = a$, for all a, b in M

(viii) $(a \vee b) + (a \wedge b) = a + b$, for all a, b in M

(ix) $[(a - b) \vee 0] + a \wedge b = a$, for all a, b in M

(x) $(a \vee b) - (a \wedge b) = (a - b) \vee (b - a)$,for all a, b in M

(xi) $a - (b - c) \leq (a - b) + c$ and $(a + b) - c \leq (a - c) + b$, for all a, b, c in M.

(xii) If $a \wedge b = 0 = a \wedge c$, then $a \wedge (b + c) = 0$, for all a, b, c in M.

Definition 1.3. Let M be a ℓ-module over R, $I \subseteq M$ and $I \neq \phi$. Then I is called a ℓ-ideal of M, if it satisfies the following properties

(i) $x, y \in I \Rightarrow x - y, x \vee y, x \wedge y \in I$

(ii) $\alpha \in R, x \in I \Rightarrow \alpha x \in I, \alpha > 0, \alpha$ is unit.

(iii) $x \in I$, $y \in M$ and $|y| < |x| \Rightarrow y \in I$

Definition 1.4. A ℓ-ideal D of a ℓ-module M is called a distributive ℓ-ideal if and if $D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y)$, for all $X, Y \in I(M)$
where $I(M)$ is the set of all ℓ-ideals of a ℓ-module.

2. Distributive Convex ℓ-Submodule

Definition 2.1. A ℓ-submodule D of a ℓ-module is said to be convex ℓ-submodule if $a, b \in D, c \in M$ and $a \leq c \leq b$ implies $c \in D$.

Definition 2.2. A convex ℓ-submodule generated by a subset A of a ℓ-module M will be denoted by $< A >$. For any two non-empty subsets A and B of a ℓ-module M we define

$$\begin{aligned} A \vee B &= < \{a \vee b / a \in A, b \in B\} > \\ A \wedge B &= < \{a \wedge b / a \in A, b \in B\} > \\ A + B &= < \{a + b / a \in A, b \in B\} > \\ \alpha A &= < \{\alpha a / a \in A, \alpha \in R\} > \end{aligned}$$

That is $A \vee B$, $A \wedge B$, $A + B$ and αA are the convex ℓ- submodule of M generated by the elements $a \vee b$, $a \wedge b$, $a + b$ and αa, for all $a \in A, b \in B$ and $\alpha \in R, \alpha > 0, \alpha$ is unit respectively.

Definition 2.3. A convex ℓ-submodule D is called distributive if

$$\begin{aligned} < D, X \wedge Y > &= < D, X > \wedge < D, Y > \\ < D, X \vee Y > &= < D, X > \vee < D, Y > \\ < D, X + Y > &= < D, X > + < D, Y > \\ < D, \alpha X > &= \alpha < D, X > \end{aligned}$$

hold for any pair of convex ℓ-submodules X, Y of M and $\alpha \in R, \alpha > 0$ whenever neither $D \cap X$ nor $D \cap Y$ are empty.

Theorem 2.4. For each d in a Browerian Algebra A, $\{0, d\}$ is a distributive convex ℓ-submodule of A.

Theorem 2.5. A ℓ-ideal D of a ℓ-module M is distributive iff it is a distributive convex ℓ-submodule of M.

Theorem 2.6. A dual ℓ-ideal D' of a ℓ-module is distributive if and only if it is a distributive convex ℓ-submodule of M.

Corollary 2.7. If a ℓ-ideal D of a ℓ-module M is standard then it is distributive convex ℓ-submodule of M.

Proof. D is a standard ℓ-ideal.
$\Rightarrow D$ is a distributive ℓ-ideal, by characterization theorem for standard ℓ-ideal.
$\Rightarrow D$ is a distributive convex ℓ-submodule by Theorem 6.2. ■

Corollary 2.8. If a ℓ-ideal D of a ℓ-module M is neutral then it is a distributive convex ℓ-submodule of M.

Theorem 2.9. Let D be a convex ℓ-submodule of a ℓ-module M.If x, y in M such that $x \vee t = y \vee t, x \wedge s = y \wedge s, x + u = y + u, \alpha x = \alpha y$, for some $s, t, u \in D$, $\alpha \in R$, then $< D, \{x\} > \ = \ < D, \{y\} >$.

Theorem 2.10. Let M be a ℓ-module and D a distributive convex ℓ-submodule of M. Then
If D satisfies property (P) where (P) is

$$\left.\begin{array}{lcl} < D, X \vee Y > & = & < D, X > \vee < D, Y > \\ < D, X \wedge Y > & = & < D, X > \wedge < D, Y > \\ < D, X + Y > & = & < D, X > + < D, Y > \\ < D, \alpha X > & = & \alpha < D, X > \end{array}\right\} (P)$$

for all single element convex ℓ-submodule X, Y of M and $\alpha \in R$, then the binary relation θ_D on M is defined by

"$x \equiv y(\theta_D)$ iff

$(x \wedge y) \wedge s = (x \vee y) \wedge s$

$(x \wedge y) \vee t = (x \vee y) \vee t$

$(x \wedge y) + u = (x \vee y) + u$

$\alpha(x \wedge y) = \alpha(x \vee y)$ for suitable s, t, u in D" is a congruence relation.

Theorem 2.11. If D is a convex ℓ-submodule of M such that the relation θ_D defined in the above theorem is a congruence relation, then D is a distributive convex ℓ-submodule of M.

Corollary 2.12. If D is a distributive convex ℓ-submodule of M, then D is a congruence class by the congruence relation θ_D provided D satisfy (P).

Corollary 2.13. If D_1 and D_2 are two distributive convex ℓ-submodules of M satisfying the property (P) then $D_1 \cap D_2$ is either a distributive convex ℓ-submodule or it is empty.

Theorem 2.14. Let $f : x \longrightarrow x'$ be a homomorphism of M onto M' and let D be a distributive convex ℓ-submodule of M.Then the homomorphic image D' of D is a distributive convex ℓ-submodule of M'.

Theorem 2.15. First Isomorphism Theorem:
Let M be a ℓ-module, D a distributive convex ℓ-submodule and I an ℓ-ideal of M such that $D \leq I$. Then I is a distributive convex ℓ-submodule in M if and only if I/D is a convex ℓ-submodule in M/D and in this case the isomorphism $M/I \cong (M/D)/(I/D)$ hold.

Theorem 2.16. Second Isomorphism Theorem:
Let M be a ℓ-module D the distributive convex ℓ-submodule and I a ℓ-ideal of M such that $I \cap D \neq \phi$. Then $I \cap D$ is a distributive convex ℓ-submodule of I and $< I, D > /D \cong I/I \cap D$.

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