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Simultaneous Approximation by modified beta operators

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Abstract

In the present paper we establish direct and inverse theorems in simultaneous approximation in terms of Ditzian-Totik weighted modulus of smoothness for a sequence of the positive linear operators introduced by Gupta and Ahmad. In particular for $\lambda = 0$, we obtain the corresponding results in terms of the usual modulus of smoothness.

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1. Introduction

It is observed that the summation type operators are not suitable for L_p- approximation. For this reasons the summation type operators are appropriately modified to become L_p-approximation methods. Durrmeyer [6] modified the Bernstein polynomials so that to approximate the Lebesgue integrable functions on [0, 1], as follows

$$M_n(f,x) = (n+1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)\,dt,$$

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$p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$. These operators have been extensively studied by Derriennic [3], and several other researchers (see [6],[8],[12] and the references therein). Motivated by this integral modification of Bernstein polynomials and subsequent work on Bernstein Durrmeyer operators, Gupta and Ahmad [9] introduced following hybrid type operators in order to approximate Lebesgue integrable functions on the interval $[0,\infty)$:

$$B_n(f,x) = \frac{n-1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} p_{n,k}(t) f(t)\, dt,$$

where $p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$, $b_{n,k}(x) = \frac{1}{B(k+1,n)} \frac{x^k}{(1+x)^{n+k+1}}$ by combining the weights of two different operators, namely Beta and Baskakov operators. Gupta and Ahmad studied these operators for the rate of convergence for functions of bounded variation.

The $K-$functional $\overline{K}_{2,\varphi^\lambda}(f^{(s)},t)$ and the corresponding Ditzian-Totik modulus of smoothness $\omega^2_{\varphi^\lambda}(f^{(s)},t)$(cf.[5]) we shall use in our study are defined as:

Let $f^{(s)} \in C_B[0,\infty)$, the class of bounded and continuous functions on $[0,\infty)$, $0 \leqslant \lambda \leqslant 1$ $\varphi(x) = \sqrt{x(1+x)}$, then

$$\omega^2_{\varphi^\lambda}(f^{(s)},t) = \sup_{0<h\leqslant t} \sup_{x+2h\varphi^\lambda(x)\geqslant 0} \left\| \overrightarrow{\Delta}^2_{h\varphi^\lambda(x)} f^{(s)}(t) \right\|,$$

where the second order forward difference of the function $f^{(s)}$ at a point x is given by

$$\overrightarrow{\Delta}^2_{h\varphi^\lambda(x)} f^{(s)}(x) = \begin{cases} \sum_{j=0}^{2} (-1)^{2-j} \binom{2}{j} f^{(s)}\big(x + jh\varphi^\lambda(x)\big) \\ \text{if } x, x+2h\varphi^\lambda(x) \in [0,\infty) \\ 0, \quad \text{otherwise} \end{cases}$$

and

$$\overline{K}_{2,\varphi^\lambda}(f^{(s)},t^2) = \inf_{g\in W_{2,\lambda}} \left\{ \|f^{(s)} - g\| + t^2\|\varphi^{2\lambda} g''\| + t^4\|g''\| \right\}$$

where the class $W_{2,\lambda}$ is given by $\left\{g : \|\varphi^{2\lambda}g''\| < \infty, g' \in AC_{loc}(0,\infty)\right\}$ and $\varphi(x) = \sqrt{x(1+x)}$ is an admissible weight function of Ditzian-Totik modulus of smoothness. It is easy to see that $\varphi^\lambda(x)$ satisfies properties (I)-(III)p.8 [5]. Moreover, the following equivalence is well known (p. 11,[5])

$$\omega^2_{\varphi^\lambda}(f^{(s)},t) \sim \overline{K}_{2,\varphi^\lambda}(f^{(s)},t^2).$$

The class of bounded Lebesgue integrable functions and r times continuously differentiable functions on the interval $[0,\infty)$ are denoted by $L_B[0,\infty)$ and $C^r[0,\infty)$

respectively. Further, $\mathbb{N}^0$ is the set of the non-negative integers and the constant M is not the same at each occurrence. Section 2 provides some lemmas and corollaries regarding the moment estimates and the Bernstein type inequalities. Finally, the direct and inverse results are the contents of Section 3.

2. Some Lemmas

The contents of this section are some auxiliary results and lemmas which will be used in our main theorems.

Lemma 2.1. For the functions $A_{m,n}(x)$ given by

$$A_{m,n}(x) \equiv \sum_{k=0}^{\infty} \left(\frac{k}{n+1} - x \right)^m b_{n,k}(x),$$

we have:

(a) $A_{0,n}(x) = n,\ A_{1,n}(x) = 0;$

(b) $(n+1)A_{m+1,n}(x) = \varphi^2(x) \left\{ A'_{m,n}(x) + m\, A_{m-1,n}(x) \right\}$, where $m \geqslant 1,\ x \in [0, \infty)$ and $\varphi^2(x) = x(1+x)$;

(c)

$$A_{2m,n}(x) \leqslant C_m\, n^{-m+1} \left(\delta_n^{2m}(x) + n^{-1} \right),$$

for all $m \in \mathbb{N}^0$, where C_m is a constant that depends on m and $\delta_n(x) = \varphi(x) + \dfrac{1}{\sqrt{n}}$.

Proof. (2.1) and (2.1) follow from direct calculations and (2.1) follows by an induction on m with the help of recurrence relation (2.1) and using the equivalencies:

$$\delta_n^2(x) \sim \begin{cases} \dfrac{1}{\sqrt{n}} \text{ for } x \in \left[0, \dfrac{1}{n}\right] = E_n \\ \varphi(x) \text{ for } x \in \left(\dfrac{1}{n}, \infty\right) \end{cases}$$

■

Following is a Lorentz type lemma:

Lemma 2.2. [9] There exist polynomials $q_{i,j,r}(x)$ independent of n and k such that

$$\varphi^{2r}(x) \frac{d^r}{dx^r} b_{n,k}(x) = \sum_{\substack{2i+j \leqslant r \\ i,j \geqslant 0}} n^i [k - (n+1)x]^j q_{i,j,r}(x)\, b_{n,k}(x).$$

Lemma 2.3. [1] Let Ω be monotone increasing on $[0, c]$. Then $\Omega(t) = O(t^{\alpha}), t \to 0+$, if for some $0 < \alpha < r$ and all $h, t \in [0, c]$

$$\Omega(h) < M\left[t^{\alpha} + (h/t)^{r}\Omega(t)\right].$$

Lemma 2.4. If f is s times $(s = 1, 2, 3, \dots)$ differentiable on $[0, \infty)$, then we have

$$B_n^{(s)}(f, x) = \frac{(n-s-1)!(n+s-1)!}{n!(n-2)!}\sum_{k=0}^{\infty} b_{n+s,k}(x)\int_0^{\infty} p_{n-s,k+s}(t) f^{(s)}(t)\, dt.$$

Proof. The proof is similar to Lemma 2.2[2]. ■

We make use of the Lemma 2.4 to define the operators $B_{n;s}(f, x)$ as follows

$$B_{n;s}(f, x) = \frac{(n-s-1)!(n+s-1)!}{n!(n-2)!}\sum_{k=0}^{\infty} b_{n+s,k}(x)\int_0^{\infty} p_{n-s,k+s}(t) f(t)\, dt.$$

Obviously, $B_n^{(s)}(f, x) = B_{n;s}(f^{(s)}, x)$ and $B_{n;s}$ are linear positive operators.

Lemma 2.5. For $m \in \mathbb{N}^0$, if we define the $m-$th order moment for the operators $B_{n;s}$ by

$$U_{n,m}(x) = B_{n;s}\left((t-x)^m, x\right)$$

then

$$U_{n,0}(x) = \frac{(n+s)!(n-s-2)!}{(n)!(n-2)!}, \quad U_{n,1}(x) = \frac{(n+s)!(n-s-3)!}{(n)!(n-2)!}\big((2s+3)x + (s+1)\big)$$

and there holds the recurrence relation

$$\begin{aligned}(n-s-m-2)U_{n,m+1}(x) &= \varphi^2(x)\big(U'_{n,m}(x) + 2mU_{n,m-1}(x)\big)\\ &+ \big((2s+2m+3)x + (m+s+1)\big)U_{n,m}(x).\end{aligned}$$

Proof. $U_{n,0}(x)$ and $U_{n,1}(x)$ are obtained by direct calculations. In order to establish the recurrence relation, we make use of $\varphi^2(x)b'_{n,k}(x) = \big(k-(n+1)x\big)b_{n,k}(x)$ and $\varphi^2(x)p'_{n,k}(x) = \big(k-nx\big)p_{n,k}(x)$. Thus, we get

$$\begin{aligned}&\varphi^2(x)U'_{n,m}(x) + m\varphi^2(x)U_{n,m-1}(x)\\ &= \frac{(n-s-1)!(n+s-1)!}{n!(n-2)!}\sum_{k=0}^{\infty}\varphi^2(x)b'_{n+s,k}(x)\int_0^{\infty} p_{n-s,k+s}(t)(t-x)^m\, dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{(n-s-1)!(n+s-1)!}{n!(n-2)!} \sum_{k=0}^{\infty} b_{n+s,k}(x) \\
&\quad \times \int_0^{\infty} \big(k-(n+s+1)x\big) p_{n-s,k+s}(t)(t-x)^m\,dt \\
&= \frac{(n-s-1)!(n+s-1)!}{n!(n-2)!} \int_0^{\infty} \big[\big(k+s-(n-s)t\big)-(1+2s)(t-x) \\
&\quad -\big(s+(1+2s)x\big)\big] p_{n-s,k+s}(t)(t-x)^m\,dt + (n+s+1)U_{n,m+1}(x) \\
&= \frac{(n-s-1)!(n+s-1)!}{n!(n-2)!} \sum_{k=0}^{\infty} b_{n+s,k}(x) \int_0^{\infty} \varphi^2(t) p'_{n-s,k+s}(t)(t-x)^m\,dt
\end{aligned}
$$

$$
-\big(s+(1+2s)x\big)U_{n,m}(x) - (1+2s)U_{n,m+1}(x) + (n+s+1)U_{n,m+1}(x) \tag{2.1}
$$

The integral I, say in (2.1) can be written as

$$
\begin{aligned}
I = \frac{(n-s-1)!(n+s-1)!}{n!(n-2)!} \sum_{k=0}^{\infty} b_{n+s,k}(x) \int_0^{\infty} \big[\varphi^2(x)+(1+2x)(t-x)+(t-x)^2\big] \\
\times\, p'_{n-s,k+s}(t)(t-x)^m\,dt
\end{aligned}
$$

Now, integration by parts gives

$$
I = -m\varphi^2(x)U_{n,m-1}(x) - (m+1)(1+2x)U_{n,m}(x) - (m+2)U_{n,m+1}(x). \tag{2.2}
$$

Using (2.2) in (2.1) and rearranging the terms we get the required relation. ■

Corollary 2.6. From Lemma 2.5, it follows that

$$
\begin{aligned}
U_{n,2}(x) &= \frac{(n-s-1)!(n+s-1)!}{n!(n-2)!}\big[2(n+s^2+8s+7)\varphi^2(x) - 2(s+2)x \\
&\quad +(s+1)(s+2)\big] \\
&\leqslant M\big(n^{-1/2}\varphi(x)\big)^2 \leqslant Mn^{-1}\delta_n^2(x).
\end{aligned}
$$

Further, by an induction on m it follows easily that

$$
\begin{aligned}
U_{n,2m}(x) &= \sum_{i=0}^{m} q_{i,2m}\left(\frac{\varphi^2(x)}{n}\right)^{m-i} n^{-2i}; \\
U_{n,2m+1}(x) &= \sum_{i=0}^{m} q_{i,2m+1}\left(\frac{\varphi^2(x)}{n}\right)^{m-i} n^{-2i-1};
\end{aligned}
$$

and

$$U_{n,2m}(x) = B_{n;s}\Big((t-x)^{2m}, x\Big) \leqslant M\big(n^{-1/2}\delta_n(x)\big)^{2m},$$

where the numbers $q_{i,2m}$ and $q_{i,2m+1}$ are independent of x and are uniformly bounded in n.

Our next result is a Bernstein type lemma which we shall use in inverse theorem.

Lemma 2.7. If $f \in L_B[0,\infty)$, $f^{(r-1)} \in AC_{loc}(0,\infty)$ and $r \in N$ then, there hold the inequality:

$$\big|B_{n,s}^{(r)}(f,x)\big| \leqslant M\varphi^{-\lambda r}(x)\big\|\varphi^{\lambda r} f^{(r)}\big\|,$$

where $M = M(r)$ is a constant that depends on r but is independent of f and n.

Proof. From Lemma 2.5 it follows that $B_{n,s}\big((t-x)^s, x\big)$ are polynomials in x of degree s. Therefore, $B_{n,s}^{(r)}\big((t-x)^s, x\big) = 0$ for $s < r$. By the assumption we can write

$$f(t) = \sum_{s=0}^{r-1} \frac{f^{(s)}(x)(t-x)^s}{s!} + R_r(f,t;x),$$

where $R_r(f,t;x) = \dfrac{1}{(r-1)!}\displaystyle\int_x^t (t-u)^{r-1} f^{(r)}(u)\,du.$

$$B_{n,s}^{(r)}(f,x) \;=\; B_{n,s}^{(r)}\Big(R_r(f,t;x), x\Big),$$

Making use of

$$\left|\int_x^t (t-u)^{r-1} f^{(r)}(u)\,du\right| \leqslant \frac{|t-x|^r\,\|\varphi^{\lambda r} f^{(r)}\|}{x^{\lambda r/2}}\left(\frac{1}{(1+x)^{\lambda r/2}} + \frac{1}{(1+t)^{\lambda r/2}}\right)$$

we get,

$$\begin{aligned}
&|B_{n,s}^{(r)}(f,x)|\\
&\leqslant \frac{\|\varphi^{\lambda r} f^{(r)}\|}{(r-1)!}\frac{(n-s-1)!(n+s-1)!}{n!(n-2)!}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}\sum_{k=0}^{\infty} n^i |k-(n+s+1)x|^j \times\\
&\times \frac{|q_{i,j,r}(x)|}{\varphi^{2r}(x)} b_{n+s,k}(x)\Bigg[\int_0^\infty p_{n-s,k+s}(t)\frac{|t-x|^r}{\varphi^{\lambda r}(x)}\,dt +\\
&+ \int_0^\infty p_{n-s,k+s}(t)\frac{|t-x|^r}{x^{\lambda r/2}}\frac{1}{(1+t)^{\lambda r/2}}\,dt\Bigg]\\
&= I_1 + I_2 \text{ say.}
\end{aligned}$$

We write $M = \sup\limits_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \|q_{i,j,r}(x)\|$ and make use of Hölder's inequalities for integration and summation, the value $\int\limits_0^\infty p_{n-s,k+s}(t) = \dfrac{1}{n-s-1}$ and Lemma 2.1, Lemma 2.5 to obtain following estimates

$$\begin{aligned} I_1 &\leqslant \frac{M\left\|\varphi^{\lambda r} f^{(r)}\right\|}{(r-1)!\varphi^{2r+2\lambda}(x)} \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \left(\sum_{k=0}^{\infty}\Big(\frac{k}{n+s+1}-x\Big)^{2j} b_{n+s,k}(x)\right)^{\frac{1}{2}} \frac{(n+s+1)^j}{\sqrt{n-s-1}} \\ &\times\ (n+s)^i \left(\frac{(n-s-1)!(n+s-1)!}{n!(n-2)!}\sum_{k=0}^{\infty} b_{n+s,k}(x)\int\limits_0^\infty (t-x)^{2r} p_{n-s,k+s}(t)\,dt\right)^{\frac{1}{2}} \\ &\leqslant M\frac{\left\|\varphi^{\lambda r} f^{(r)}\right\|}{(r-1)!\varphi^{2r+2\lambda}(x)} \frac{1}{\sqrt{n-s-1}} \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} (n+s)^i \Big(n^{-j+1}\delta_n^{2j}(x)\Big)^{\frac{1}{2}} n^{-r/2}\delta_n^r(x) \\ &\leqslant M\varphi^{-\lambda r}(x)\left\|\varphi^{\lambda r} f^{(r)}\right\|, \end{aligned}$$

where we have used the equivalence $\delta_n(x) \sim \dfrac{1}{\sqrt{n}}$ for $x \in E_n$ and for $x \in E_n^c$, $\delta_n(x) \sim \varphi(x)$. Now it follows by direct calculations that $\int\limits_0^\infty p_{n-s,k+s}(t)(1+t)^{-r\lambda}\,dt \leqslant M(1+x)^{-r\lambda}$. Therefore, we get

$$\begin{aligned} I_2 &\leqslant \varphi^{\lambda r}\frac{\|\varphi^{\lambda r} f\|}{x^{r\lambda/2}} \frac{(n-s-1)!(n+s-1)!}{n!(n-2)!} \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \sum_{k=0}^{\infty} n^i |k-(n+s+1)x|^j \frac{|q_{i,j,r}(x)|}{\varphi^{2r}(x)} \\ &\times\ b_{n+s,k}(x)\int\limits_0^\infty p_{n-s,k+s}(t)|t-x|^r(1+t)^{-r\lambda/2}\,dt \\ &\leqslant M\frac{(n-s-1)!(n+s-1)!}{n!(n-2)!} \frac{\|\varphi^{\lambda r} f^{(r)}\|}{\varphi^{2r}(x)x^{r\lambda/2}} \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} \sum_{k=0}^{\infty} n^i|k-(n+s+1)x|^j \times \\ &\times\ b_{n+s,k}(x)\Big(\int\limits_0^\infty p_{n-s,k+s}(t)(t-x)^{2r}\,dt\Big)^{\frac{1}{2}}\Big(\int\limits_0^\infty p_{n-s,k+s}(t)(1+t)^{-r\lambda}\,dt\Big)^{\frac{1}{2}} \end{aligned}$$

$$\leqslant \quad M\frac{\|\varphi^{\lambda r}f^{(r)}\|}{\varphi^{(2+\lambda)r}(x)}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}(n+s)^i(n+s+1)^j\left(\sum_{k=0}^{\infty}\left(\frac{k}{n+s+1}-x\right)^{2j}b_{n+s,k}(x)\right)^{\frac{1}{2}}$$

$$\times \quad \left(\frac{(n-s-1)!(n+s-1)!}{n!(n-2)!}\sum_{k=0}^{\infty}b_{n+s,k}(x)\int\limits_0^{\infty}(t-x)^{2r}p_{n-s,k+s}(t)\,dt\right)^{\frac{1}{2}}$$

$$\leqslant \quad M\left\|\varphi^{\lambda r}f^{(r)}\right\|.$$

Lemma 2.8. If $f\in L_B[0,\infty)$ and $r\in N$ then, there hold the inequalities:

$$\left|B_{n,s}^{(r)}(f,x)\right|\leqslant Mn^{r/2}\delta_n^r(x)\varphi^{-2r}(x)\left\|f\right\|,$$

where $M=M(r)$ is a constant that depends on r but is independent of f and n.

The proof of is similar to Lemma 2.7. ■

3. Main Results

In this section we establish the direct and inverse theorems in simultaneous approximation by the operators $B_n(f,x)$.

Theorem 3.1. If $f\in L_B[0,\infty)$, $f^{(r-1)}\in AC_{loc}(0,\infty)$, $0\leqslant\lambda\leqslant 1$, $0<\alpha<2$ and $\varphi(x)=\sqrt{x(1+x)}$ then, we have

$$\left|B_n^{(r)}\big(f,x\big)-f^{(r)}(x)\right|\leqslant M\omega_{\varphi^\lambda}^2\left(f^{(r)},n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)\right)+\omega\big(f^{(r)},\eta\big),$$

where $\eta=\sup\limits_{0\leqslant x<\infty}B_{n,r}(t-x,x)$.

Proof. Let us take $g_{n,x,\lambda}=g\in W_{2,\lambda}$ such that

$$\|f^{(r)}-g\|+\left(n^{-\frac{1}{2}}\varphi_n^{1-\lambda}(x)\right)^2\|\varphi^{2\lambda}g''\|\leqslant 2\overline{K}_{2,\varphi^\lambda}\left(f^{(r)},\left(n^{-\frac{1}{2}}\varphi_n^{1-\lambda}(x)\right)^2\right). \quad (3.1)$$

We introduce the auxiliary operators $\widehat{B}_{n,r}$ defined by

$$\widehat{B}_{n,r}(f,x)=\frac{1}{C_{n,r}}\left[B_{n,r}(f,x)-f^{(r)}(x+z)+f^{(r)}(x)\right], \quad (3.2)$$

where $z=B_{n,r}(t-x,x)$, $C_{n,r}=B_{n,r}(1,x)=\dfrac{(n+r)!(n-r-2)!}{n!(n-2)!}$ and $x\in[0,\infty)$. The operators $\widehat{B}_{n,r}$ are obviously linear and preserve the linear functions. Moreover, it follows from direct calculation that $\widehat{B}_{n,r}(1,x)=1$, $\widehat{B}_{n,r}(t-x,x)=0$ and

$$\begin{aligned}B_n^{(r)}(f,x)-f(x) \quad &= \quad C_{n,r}\Big[\widehat{B}_{n,r}(f^r-g,x)+\widehat{B}_{n,r}(g,x)-g(x)\\ &\qquad +C_{n,r}g(x)-f^r(x)+f(x+z)-f(x)\Big]\end{aligned}$$

Hence, in view of the limit $C_{n,r} \to 1$ as $n \to \infty$, we get

$$|B_n^{(r)}(f,x) - f(x)| \leqslant 2\|f^{(r)} - g\| + |\widehat{B}_{n,r}(g,x) - g(x)| + |f^{(r)}(x+z) - f^{(r)}(x)| \quad (3.3)$$

Using the smoothness of g, and in view of $\widehat{B}_{n,r}(t-x,x) = 0$, we get

$$|\widehat{B}_{n,r}(g,x) - g(x)| \leqslant \left|B_{n,r}\big(R_2(g,t,x)\right| + \left|\int_x^{x+z}(x+z-u)g''(u)du\right|.$$

where $R_2(g,t,x) = \int_x^t (t-u)g''(u)du$. Now following holds (see[5]p141.)

$$\begin{aligned}|R_2(g,t,x)| &\leqslant \frac{|t-x|}{x^\lambda}\left(\frac{1}{(1+x)^\lambda} + \frac{1}{(1+t)^\lambda}\right)\left|\int_x^t \varphi^{2\lambda}(u)\big|g''(u)\big|du\right| \\ &\leqslant \|\varphi^{2\lambda}g''\|(t-x)^2\left(\frac{1}{x^\lambda(1+x)^\lambda} + \frac{1}{x^\lambda(1+t)^\lambda}\right).\end{aligned}$$

Also it can be verified (cf.[7])that $B_{n,r}\big((1+t)^{-m},x\big) \leqslant C(1+x)^{-m}$. Therefore, we get

$$\begin{aligned}&B_{n,r}\big(R_2(g,t,x)\big) \\ &\leqslant \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)}B_{n,r}\Big((t-x)^2,x\Big) + \frac{\|\varphi^{2\lambda}g''\|}{x^\lambda}B_{n,r}\Big(\frac{(t-x)^2}{(1+t)^\lambda},x\Big) \\ &\leqslant \frac{\|\varphi^{2\lambda}g''\|}{\varphi^{2\lambda}(x)}B_{n,r}\big((t-x)^2,x\big) + \frac{\|\varphi^{2\lambda}g''\|}{x^\lambda}\sqrt{B_{n,r}\big((t-x)^4,x\big)}\sqrt{B_{n,r}\big((1+t)^{-2\lambda},x\big)} \\ &\leqslant M\|\varphi^{2\lambda}g''\|\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2.\end{aligned}$$

Since, $z \leqslant C\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2$ for all values of x, therefore we obtain

$$\left|\int_x^{x+z}(x+z-u)g''(u)du\right| \leqslant \big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^4\|g''\|.$$

Collecting these estimates, we get

$$|\widehat{B}_{n,r}(g;x) - g(x)| \leqslant C\|\varphi^{2\lambda}g''\|\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2 + \big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^4\|g''\|.$$

Therefore, we have

$$\begin{aligned}&|B_n^{(r)}(f,x) - f(x)| \\ &\leqslant C\left(\|f^{(r)} - g\| + \|\varphi^{2\lambda}g''\|\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2 + \big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^4\|g''\|\right) \\ &\quad + |f^{(r)}(x+z) - f^{(r)}(x)|.\end{aligned}$$

This in view of equivalence of $\overline{K}_{2,\varphi^\lambda}(f,t^2)$ and $\omega^2_{\varphi^\lambda}(f,t)$ gives

$$\begin{aligned}|B_{n,r}(f,x)-f(x)| &\leqslant C\overline{K}_{2,\varphi^\lambda}\big(f,\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)^2\big)+\omega(f^{(r)},z)\\ &\leqslant C\omega^2_{\varphi^\lambda}\big(f,\big(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\big)+\omega\big(f^{(r)},\eta\big).\end{aligned}$$

This completes the proof of the theorem. ■

Theorem 3.2. [Inverse] Let $f\in L_B[0,\infty)$, $0\leqslant\lambda\leqslant 1$, $0<\alpha<2$ and $\varphi(x)=\sqrt{x(1+x)}$. Then, there holds the implication:

$$\left|B_n^{(r)}\big(f,x\big)-f^{(r)}(x)\right| = O\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha \Rightarrow \omega^2_{\varphi^\lambda}(f,x)=O\,(t)^\alpha\,. \quad (3.4)$$

Proof. We have

$$\begin{aligned}\left|\overrightarrow{\Delta}^2_{h\varphi^\lambda(x)}f(x)\right| &\leqslant \left|\overrightarrow{\Delta}^2_{h\varphi^\lambda(x)}\left(f^{(r)}(x)-B_n^{(r)}(f,x)\right)\right|+\left|\overrightarrow{\Delta}^2_{h\varphi^\lambda(x)}B_{n,r}(f^{(r)},x)\right|\\ &\leqslant M\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha+\left|\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}B''_{n,r}(f^{(r)}-g,x+u+v)du\,dv\right|\\ &\quad+\left|\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}\int_{-\frac{h\varphi^\lambda(x)}{2}}^{\frac{h\varphi^\lambda(x)}{2}}B''_{n,r}(g,x+u+v)du\,dv\right|.\end{aligned}$$

Using Lemma 2.7, and Lemma 2.8, we obtain

$$\begin{aligned}\omega^2_{\varphi^\lambda}(f,h) &\leqslant M\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha+\eta^2\left(\varphi^{-2\lambda}\,n\,\delta_n^{-2}(x)\|f^{(r)}-g\|+\varphi^{-2\lambda}\|\varphi^{2\lambda}g''\|\right)\\ &\leqslant M\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha+\left(\frac{h}{n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)}\right)^2\\ &\quad\times\left(\|f^{(r)}-g\|+\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^2\|\varphi^{2\lambda}g''\|\right)\\ &\leqslant M\left(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\right)^\alpha+\left(\frac{h}{n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)}\right)^2\omega^2_{\varphi^\lambda}\Big(f,n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)\Big).\end{aligned}$$

Finally, using Lemma 2.3 the proof of the theorem follows. ■

Remark 3.3. Analogous to Theorem 1,[7] we can obtain the corresponding theorem for the range $0<\alpha\leqslant 1$ while for $r=1$ from Theorem 3.1 and Theorem 3.2 we obtain following theorem for the range $0<\alpha<2$:

Theorem 3.4. Let $f \in L_B[0,\infty)$, $\varphi(x) = \sqrt{x(1+x)}$, $0 \leqslant \lambda \leqslant 1$ and $0 < \alpha < 2$. Then, there holds the implication (i) $\Leftrightarrow$ (ii) in the following statements:

(i) $\left|B_n(f,x) - f(x)\right| = O\left(\dfrac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right)^{\alpha}$

(ii) $\omega_{\varphi^\lambda}(f,x) = O(x^\alpha)$.

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References

[1] Berens, H. and Lorentz, G. G., Inverse theorem for Bernstein polynomials, *Indiana Univ. Math. J.*, 21:693–708, 1972.

[2] Deo, N., Direct Result on the Durrmeyer Variant of Beta Operators, *Southeast Asian Bulletin of Mathematics*, 32:283–290, 2008.

[3] Derriennic, M. M., Sur l'approximation de functions integrable sur [0,1] par des polynomes de Bernstein modifies, *J. Approx. Theory*, 31:325–343, 1981.

[4] Ditzian, Z. On interpolation of $L_p[a,b]$ and weighted Sobolov spaces, *Pacific J. Math.*, 90:307–323, 1980.

[5] Ditzian, Z. and Totik, V., *Moduli of Smoothness*, Springer, New York, 1987.

[6] Durrmeyer, J. L., Une formule d'inversion de la transformee de Laplace: Application a la Theorie des Moments, These de 3e Cycle, Faculte des Sciences de l' Universite de Paris 1967.

[7] Gairola, A. R. and Agrawal, P. N., Direct and inverse theorems for the Bézier variant of certain summation-integral type operators, *Turk. J. Math.*, 33:1–14, 2009.

[8] Gonska, H. H. and Zhou, X. L., A global inverse theorem on simultaneous approximation by Bernstein-Durrmeyer operators, *J. Approx. Theory*, 67:284–302, 1991.

[9] Gupta, V. and Ahmad, A., Simultaneous approximation by Modified Beta operators, *Istanbul Univ. Fen. Fak. Mat. Der.*, 54:11–22, 1995.

[10] Gupta, V. and Mohapatra, R., On the rate of convergence for certain summation-integral type operators, *Math. Ineq. Appl.*, 9(3):465–472, 2006.

[11] Hewitt E. and Stromberg, K., *Real and Abstract Analysis*, McGraw-Hill, New-York, 1969.

[12] Lorentz, G. G., *Bernstein Polynomials*, Toronto Press, Toronto, 1953.

[13] Maheshweari, P., An estimation of the rate of convergence for modified beta operators, *Indian J. Pure Appl. Math.*, 34(6):927–934, 2003.

[14] Timan, A. F., *Theory of Approximation of Functions of a Real Variable* (English Translation), Dover Publications, Inc., N.Y., 1994.