Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 3, Number 4 (2011), pp. 317-325

http://www.irphouse.com

A Note On Certain Retarded Integral Inequalities

J.V. Patil

Vasantrao Naik Mahavidyalaya, Aurangabad-431001, India
E-mail: jv.patil29@gmail.com

Abstract

In this note, we generalize two retarded integral inequalities.one of these inequalities says:

If

$$w^p(t) \le h^p(t) + p\int_0^{\alpha(t)} \left\{ f(s)w(s)\left[w^{p-1}(s) + \int_0^s g(\gamma)w^{p-1}(\gamma)d\gamma \right] + q(s)w(s)ds \right\}, p > 1 \qquad (1)$$

then

$$w(t) \le \left\{ \left(h(t) + (p-1)\int_0^{\alpha(t)} q(s)ds \right) \left[1 + (p-1)\int_0^{\alpha(t)} f(s)exp\left(\int_0^s (f(\sigma) + g(\sigma))d\sigma \right) ds \right] \right\}^{\frac{1}{p-1}} \qquad (2)$$

for t $\in [0,\infty)$

under suitable conditions on functions w, α ,h,f,g and p on $[0,\infty)$.

AMS Subject Classifications: 26D15.

Keywords: integral inequality, retarded.

Introduction and Preliminaries

In [1] Wong and Yeh obtained a bound on the following inequality:

$$w^2(t) \le h^2(t) + 2\int_0^{\alpha(t)} \left\{ f(s)w(s)[w(s) + \int_0^s g(\rho)w(\rho)d\rho] + q(s)w(s) \right\} ds,$$

in the form ,

$$w(t) \le \left[h(t) + \int_0^{\alpha(t)} q(s)ds \right] exp\left\{ \int_0^{\alpha(t)} \left[f(s) + (\int_0^s g(\gamma)d\gamma)ds \right] \right\},\ t \ge 0$$

under suitable conditions on the functions f,g,h,w , α and q on $[0,\infty)$.

In [1], the authors tried to obtain the generalizations of the inequalities in [2] and did not succeed because of their incorrect proof for Theorem 2.3 . The aim of this paper is to correct the explicit bound on the inequality in Theorem 2.3 in [1] and also obtain an explicit bounds for the general versions of above inequalities proved by Wong and Yeh in [1]. To show usefulness of our results an application is also given.

For convenience, we assume throughout this paper that the following conditions hold :

(i) w,f,g,h and q $\in C([0,\infty),(0,\infty))$ with h increasing and p > 1,

(ii) $\alpha \in C^1([0,\infty),(0,\infty)), \alpha(t) \le t \; and \; \alpha'(t) \ge 0 \; on [0,\infty)$

In order to discuss our main results, we need the following lemmas.

Lemma 1.1 [1] If

$$w(t) \le h(t) + \int_0^{\alpha(t)} f(s)w(s)ds, t \in [0,\infty), \tag{3}$$

then

$$w(t) \le h(t) exp \int_0^{\alpha(t)} f(s)ds, t \in [0,\infty), \tag{4}$$

Lemma 1.2 [1] If

$$w(t) \le h(t) + \int_0^{\alpha(t)} f(s) \int_0^t g(\sigma)w(\sigma)d\sigma ds, \quad t \ge 0 \tag{5}$$

then

$$w(t) \le h(t) exp \int_0^{\alpha(t)} \left[f(s) \int_0^s g(\sigma)d\sigma \right] ds, \quad t \ge 0 \tag{6}$$

Lemma 1.3 If

$$w(t) \le w_0 + \int_0^{\alpha(t)} f(s) \left[w(s) + \int_0^s g(\sigma)w(\sigma)d\sigma \right] ds, \quad t \ge 0 \tag{7}$$

where w_0 is non-negative constant, then

$$w(t) \le w_0 \left[1 + \int_0^{\alpha(t)} f(s) \, exp \left(\int_0^s (f(\sigma) + g(\sigma))d\sigma \right) ds) \right], \quad t \ge 0 \tag{8}$$

Proof: Define a function v(t) by the right side of the equation (1.5), then we have

$w(t) \le v(t)$ such that $v(t_0) = w_0$

$$v'(t) = f(\alpha(t))w(\alpha(t))\alpha'(t) + f(\alpha(t))w(\alpha(t))\alpha'(t) \int_0^{\alpha(t)} g(\sigma)u(\sigma)d\sigma$$

$$v'(t) = f(\alpha(t))\alpha'(t)\left(w(\alpha(t)) + \int_0^{\alpha(t)} g(\sigma)w(\sigma)d\sigma \right) \tag{9}$$

Then by our assumption on w(t) we have

$$v'(t) \le f(\alpha(t))\alpha'(t)\left(v(\alpha(t)) + \int_0^{\alpha(t)} g(\sigma)v(\sigma)d\sigma \right) \tag{10}$$

Define a function m(t) given by

$$m(t) = v(\alpha(t)) + \int_0^{\alpha(t)} g(\sigma)v(\sigma)d\sigma$$

Then we have

$$m(t) \le v(t) + \int_0^{\alpha(t)} g(\sigma)v(\sigma)d\sigma \quad (as\, \alpha(t) \le t)$$

such that

$$v(t) \le m(t)\ ,\ m(t_0) = v(t_0) = w_0\ , and\ \ v'(t) \le f(\alpha(t))\alpha'(t)m(t)$$

$$m'(t) \le v'(t) + g(\alpha(t))v(\alpha(t))\alpha'(t)$$

$$m'(t) \le f(\alpha(t))\alpha'(t)m(t) + g(\alpha(t))\alpha'(t)m(\alpha(t))$$

$$m'(t) \le \left(f(\alpha(t))\alpha'(t) + g(\alpha(t))\alpha'(t) \right) m(t)$$

Solving this further we get

$$m(t) \le w_0\, exp\left(\int_0^t \left[f(\alpha(s))\alpha'(s) + g(\alpha(s))\alpha'(s) \right] ds \right)$$

Making a change of variable on right hand side of the above inequality, we get

$$m(t) \le w_0\, exp\left(\int_0^{\alpha(t)} \left[f(s) + g(s) \right] ds \right)$$

As

$$v'(t) \le f(\alpha(t))\alpha'(t)m(t)$$

Using value of m(t), integrating above equation from 0 to t and making change of variable on right hand side, we have

$$v(t) \le w_0\left[1 + \int_0^{\alpha(t)} f(s)\left(\int_0^s (f(\sigma) + g(\sigma))d\sigma \right) ds \right]$$

As $w(t) \le v(t)$, we get the desired result in (1.6).

Lemma 1.4 : If

$$w(t) \le h(t) + \int_0^{\alpha(t)} f(s)\left(w(s) + \int_0^s g(\sigma) w(\sigma) d\sigma \right) ds,\ t \in [0,\infty) \quad (11)$$

then

$$w(t) \le h(t)\left[1 + \int_0^{\alpha(t)} f(s) exp\left(\int_0^s [f(\sigma) + g(\sigma)] d\sigma \right) ds \right],\ t \in [0,\infty) \quad (12)$$

Proof : since h(t) is positive and nondecreasing, we have

$$\frac{w(t)}{h(t)} \le 1 + \int_0^{\alpha(t)} f(s) \frac{u(s)}{h(s)} ds + \int_0^{\alpha(t)} f(s)\left(\int_0^s g(\sigma) \frac{u(\sigma)}{h(\sigma)} d\sigma \right) ds$$

Applying Lemma 1.3 to above inequality, we have

$$\frac{w(t)}{h(t)} \le 1 + \int_0^{\alpha(t)} f(s) exp\left(\int_0^s [f(\sigma) + g(\sigma)] d\sigma \right) ds,$$

which further leads to

$$w(t) \le h(t)\left[1 + \int_0^{\alpha(t)} f(s) exp\left(\int_0^s [f(\sigma) + g(\sigma)] d\sigma \right) ds \right],\ t \in [0,\infty)$$

and the proof is completed.

Main Results

We now prove our main results. First we give corrected proof of Theorem 2.3 in [1]

Theorem 2.1 : If

$$w^2(t) \le h^2(t) + 2\int_0^{\alpha(t)} \left[f(s) w(s)\left(w(s) + \int_0^s g(\gamma) w(\gamma) d\gamma \right) + q(s) w(s) \right] ds, \quad (13)$$

then

$$w(t) \le \left[h(t) + \int_0^{\alpha(t)} q(s) ds \right]\left[1 + \int_0^{\alpha(t)} f(s) exp\left(\int_0^s (f(\gamma) + g(\gamma)) d\gamma \right) ds \right] \quad (14)$$

for t $\in [0,\infty)$.

Proof : For any $\varepsilon >$ 0 and any fixed T $>$ 0, it follows from (2.1) that for $0 \le t \le T$.

$$w^2(t) \le h^2(T) + \varepsilon + 2\int_0^{\alpha(t)} \left[f(s) w(s)\left[w(s) + \int_0^s g(\gamma) w(\gamma) d\gamma \right] + q(s) w(s) \right] ds$$

= K(t) (say), 0 $\le t \le T$.

Clearly K(t) is increasing, K(t) > 0 and w(t) $\le \sqrt{K(t)}$ on [0,T]. Differentiating K(t) with respect to t and using $\alpha(t) \le t$, we obtain

$$K'(t) = 2\alpha'(t)\left\{ f(\alpha(t))w(\alpha(t))\left[w(\alpha(t)) + \int_0^{\alpha(t)} g(\gamma)w(\gamma)d\gamma \right] + q(\alpha(t))w(\alpha(t))\right\}$$

$$\le 2\sqrt{K(t)}\alpha'(t)\left\{ f(\alpha(t))\left[w(\alpha(t)) + \int_0^{\alpha(t)} g(\gamma)w(\gamma)d\gamma \right] + q(\alpha(t))\right\}$$

which implies

$$\sqrt{K(t)} \le \sqrt{\varepsilon + h^2(T)} + \int_0^{\alpha(t)} q(s)ds + \int_0^{\alpha(t)} f(s)\left[w(s) + \int_0^s g(\gamma)w(\gamma)d\gamma \right]ds$$

$$\sqrt{K(t)} \le \left(\sqrt{\varepsilon + h^2(T)} + \int_0^{\alpha(t)} q(s)ds \right) + \int_0^{\alpha(t)} f(s)\left[\sqrt{K(s)} + \int_0^s g(\gamma)\sqrt{K(\gamma)}d\gamma \right]ds$$

By applying Lemma(1.4) to above inequality, we have for 0 $\le t \le$ T

$$\sqrt{K(t)} \le \left(\sqrt{\varepsilon + h^2(T)} + \int_0^{\alpha(t)} q(s)ds \right)\left[1 + \int_0^{\alpha(t)} f(s)exp\left(\int_0^s \left(f(\gamma) + g(\gamma)\right)d\gamma \right)ds \right].$$

Taking t = T and w(t) $\le \sqrt{K(t)}$, we get

$$w(t) \le \left(\sqrt{\varepsilon + h^2(T)} + \int_0^{\alpha(t)} q(s)ds \right)\left[1 + \int_0^{\alpha(t)} f(s)exp\left(\int_0^s \left(f(\gamma) + g(\gamma)\right)d\gamma \right)ds \right]$$

Letting $\varepsilon \to 0^+$ and noting T > 0 arbitrary, we obtain the desired result in 2.2
Now, we generalize the inequalities establish in [1] in the next theorems.

Theorem 2.2: If

$$w^p(t) \le h^p(t) + p\int_0^{\alpha(t)} \left[f(s)w(s)\int_0^s g(\gamma)w^{p-1}(\gamma)d\gamma + q(s)w(s) \right]ds,\ p > 1 \qquad (15)$$

then

$$w(t) \le \left\{ \left(h(t) + \int_0^{\alpha(t)} q(s)ds \right) exp\left((p-1)\int_0^{\alpha(t)} f(s)\int_0^s g(\gamma)d\gamma ds \right)\right\}^{\frac{1}{p-1}} \qquad (16)$$

for t $\in [0,\infty)$.

Proof : for any $\varepsilon > 0$ and any fixed T > 0, it follows from (2.3) that for $0 \le t \le T$

$$w^p(t) \le h^p(T) + \varepsilon + p\int_0^{\alpha(t)}\left[f(s)w(s)\int_0^s g(\gamma)w^{p-1}(\gamma)d\gamma + q(s)w(s)\right]ds = Z(t)\ (say)$$

Clearly $w^p(t) \le Z(t)$ and hence w(t) $\le (Z(t))^{1/p}$ on [0,T]. Further Z(t) is increasing and positive.Differentiating Z(t) with respect to t we get

$$Z'(t) = p\left[f(\alpha(t))w(\alpha(t))\int_0^{\alpha(t)} g(\gamma)w^{p-1}(\gamma)d\gamma + q(\alpha(t))w(\alpha(t))\right]\alpha'(t)$$

$$Z'(t) \le p\left[f(\alpha(t))(Z(\alpha(t)))^{\frac{1}{p}}\int_0^{\alpha(t)} g(\gamma)(Z(\gamma))^{\frac{p-1}{p}}d\gamma + q(\alpha(t))(Z(\alpha(t)))^{1/p}\right]\alpha'(t)$$

From this we further get

$$Z'(t) \le p(Z(t))^{1/p}\left[f(\alpha(t))\int_0^{\alpha(t)} g(\gamma)(Z(\gamma))^{\frac{p-1}{p}} + q(\alpha(t))\right]\alpha'(t)$$

$$\frac{Z'(t)}{p(Z(t))^{1/p}} \le \left[f(\alpha(t))\int_0^{\alpha(t)} g(\gamma)(Z(\gamma))^{\frac{p-1}{p}}d\gamma + q(\alpha(t))\right]\alpha'(t)$$

This implies

$$\frac{d}{dt}\left(\frac{(Z(t))^{\frac{p-1}{p}}}{(p-1)}\right) \le \left[f(\alpha(t))\int_0^{\alpha(t)} g(\gamma)(Z(\gamma))^{\frac{p-1}{p}}d\gamma + q(\alpha(t))\right]\alpha'(t)$$

By taking t = s and integrating it with respect to s from 0 to t and making a change of variable, we have

$$(Z(t))^{\frac{p-1}{p}} \le \left(\left(h^p(T)+\varepsilon\right)^{\frac{p-1}{p}} + (p-1)\int_0^{\alpha(t)} q(s)ds\right) + (p-1)\int_0^{\alpha(t)} f(s)\int_0^s g(\gamma)(Z(\gamma))^{\frac{p-1}{p}}d\gamma ds$$

Applying Lemma 1.2 in [3] to above inequality we have

$$(Z(t))^{\frac{p-1}{p}} \le \left(\left(h^p(T)+\varepsilon\right)^{\frac{p-1}{p}} + (p-1)\int_0^{\alpha(t)} q(s)ds\right) exp\left((p-1)\int_0^{\alpha(t)} f(s)\int_0^s g(\gamma)d\gamma ds\right)$$

Hence

$$Z(t) \le \left\{\left(\left(h^p(T)+\varepsilon\right)^{\frac{p-1}{p}} + (p-1)\int_0^{\alpha(t)} q(s)ds\right) exp\left((p-1)\int_0^{\alpha(t)} f(s)\int_0^s g(\gamma)d\gamma ds\right)\right\}^{\frac{p}{p-1}}$$

As $w(t) \le (Z(t))^{1/p}$, we have

$$w(t) \le \left\{ \left(\left(h^p(T)+\varepsilon\right)^{\frac{p-1}{p}} + (p-1)\int_0^{\alpha(t)} q(s)ds \right) exp\left((p-1)\int_0^{\alpha(t)} f(s)\int_0^s g(\gamma)d\gamma ds \right) \right\}^{\frac{1}{p-1}}$$

Letting $\varepsilon \to 0^+$ and nothing T $>$ 0 arbitrary, we obtain the desired result in 2.4

Remark 1 : If we put p = 2 in Theorem 2.2 we get Theorem 2.1 in [1].

Remark 2 : If we put p = 2, $h^2(t) = c^2$, c $\ge$ 0 , f(t) = 0 and $\alpha(t)$ = t in Theorem 2.2 then we get Theorem 3.4.1 in [4].

Remark 3 : If we put p = 2 and f(s) = 0 in Theorem 2.2 we get Corollary 2.2 in [1].

Remark 4 : If If we put p = 2, $h^2(t) = c^2$, $c \ge 0$, f(t) = 0, g(t)=0, in Theorem 2.2, we get corollary 1 in [3].

Theorem 2.3 : If

$$w^p(t) \le h^p(t) + p\int_0^{\alpha(t)} \left\{ f(s)w(s)\left[w^{p-1}(s) + \int_0^s g(\gamma)w^{p-1}(\gamma)d\gamma \right] + q(s)w(s)ds \right\}, p>1 \quad (17)$$

then

$$w(t) \le \left\{ \left(h(t) + (p-1)\int_0^{\alpha(t)} q(s)ds \right)\left[1 + (p-1)\int_0^{\alpha(t)} f(s)exp\left(\int_0^s (f(\sigma)+g(\sigma))d\sigma \right)ds \right] \right\}^{\frac{1}{p-1}} \quad (18)$$

for t $\in [0,\infty)$.

Proof : for any ε $>$ 0 and any fixed T $>$ 0, it follows from (2.5) that for $0 \le t \le T$.

$$w^p(t) \le h^p(T) + \varepsilon + p\int_0^{\alpha(t)} \left[f(s)w(s)\left(w^{p-1}(s) + \int_0^s g(\gamma)w^{p-1}(\gamma)d\gamma \right) + q(s)w(s) \right] ds = Z(t)\,(say)$$

Clearly Z(t) is increasing and positive and $w^p \le Z(t)$ implies w(t) $\le (Z(t))^{1/p}$ on $[0,T]$.

Differentiating Z(t) with respect to t we get

$$Z'(t) \le \left[pf(\alpha(t))(Z(\alpha(t)))^{\frac{1}{p}}\left((Z(t))^{\frac{p-1}{p}} + \int_0^s g(\gamma)(Z(\gamma))^{\frac{p-1}{p}} d\gamma \right) + q(\alpha(s))(Z(\alpha(t)))^{\frac{1}{p}} \right]\alpha'(t)$$

$$\frac{Z'(t)}{p(Z(t))^{1/p}} \le \left[f(\alpha(t))\left((Z(t))^{\frac{p-1}{p}} + \int_0^s g(\gamma)(Z(\gamma))^{\frac{p-1}{p}} d\gamma \right) + q(\alpha(s)) \right] \alpha'(t)$$

$$\frac{d}{dt}\left(\frac{(Z(t))^{\frac{p-1}{p}}}{p-1} \right) \le \left[f(\alpha(t))\left((Z(t))^{\frac{p-1}{p}} + \int_0^s g(\gamma)(Z(\gamma))^{\frac{p-1}{p}} d\gamma \right) + q(\alpha(s)) \right] \alpha'(t)$$

Setting t = s and integrating from 0 to t and making change of variables in the above inequality, we have

$$(Z(t))^{\frac{p-1}{p}} \le \left[\left(h^p(T)+\varepsilon\right)^{\frac{p-1}{p}} + (p-1)\int_0^{\alpha(t)} q(s)ds \right]$$

$$+(p-1)\int_0^{\alpha(t)} f(s)\left[(Z(s))^{\frac{p-1}{p}} + \int_0^s g(\gamma)(Z(\gamma))^{\frac{p-1}{p}} d\gamma \right] ds$$

By applying Lemma 1.4 to the above inequality we get

$$(Z(t))^{\frac{p-1}{p}} \le \left[\left(h^p(T)+\varepsilon\right)^{\frac{p-1}{p}} + (p-1)\int_0^{\alpha(t)} q(s)ds \right]$$

$$\times\left[1+(p-1)\int_0^{\alpha(t)} f(s)\, exp\left(\int_0^s (f(\sigma+g(\sigma))d\sigma \right) ds \right]$$

This further implies

$$Z(t) \le \left[\left(h^p(T)+\varepsilon\right)^{\frac{p-1}{p}} + (p-1)\int_0^{\alpha(t)} q(s)ds \right]^{\frac{p}{p-1}}$$

$$\times\left[1+(p-1)\int_0^{\alpha(t)} f(s)\, exp\left(\int_0^s (f(\sigma+g(\sigma))d\sigma \right) ds \right]^{\frac{p}{p-1}}$$

As $w(t) \le (Z(t))^{\frac{1}{p}}$,

$$w(t) \le \left[\left(h^p(T)+\varepsilon\right)^{\frac{p-1}{p}} + (p-1)\int_0^{\alpha(t)} q(s)ds \right]^{\frac{1}{p-1}}$$

$$\times\left[1+(p-1)\int_0^{\alpha(t)} f(s)\, exp\left(\int_0^s (f(\sigma+g(\sigma))d\sigma \right) ds \right]^{\frac{1}{p-1}}$$

Letting $\varepsilon \to 0^+$ and noting T > 0 arbitrary, we obtain the desired result in (2.6)

Remark 1 : If we put p = 2 in Theorem 2.3 we get Theorem 2.3 in [1].

Remark 2 : If we put p = 2 and g(t) = 0 in Theorem 2.3 we get corollary 2.4 in [1].

Remark 3 : If we put p = 2 and $h(t) = c$ in Theorems 2.2 and 2.3 we obtained results in [2].

An Application

Consider the delay integral equation

$$w^p(t) = h^p(t) + p\int_0^{\alpha(t)}\left[w(s)M\left(s, x(s), \int_0^s N(s,\rho,x(\rho)d\rho \right) + q(s)w(s) \right]ds \qquad (19)$$

Assume that

$$\left|(M(t,u,v)\right| \le f(t)\,|\,v\,|, \left|(N(t,s,u)\right| \le g(t)\,|\,u\,|^{p-1} \qquad (20)$$

where f,g,h, α are defined as in Theorem (2.2).From equation (3.1) and (3.2) we obtain

$$\left|w(t)\right|^p \le h^p(t) + p\int_0^{\alpha(t)}\left[f(s)w(s)\int_0^s g(\gamma)(w(\gamma))^p d\gamma + q(s)w(s) \right]ds$$

Now applying theorem (2.2) to the above inequality, we get an explicit bound on the unknown function w(t) as

$$\left|w(t)\right| \le \left\{ \left(h(t)^p + (p-1)\int_0^{\alpha(t)} q(s)ds \right) exp\left[(p-1)\int_0^{\alpha(t)} f(s)\left(\int_0^s g(\gamma)d\gamma \right)ds \right] \right\}^{\frac{1}{p-1}} \qquad (21)$$

References

[1] Wong, F. H., and Yeh,C.C., 2008, *"A Note of Retarded OuYang Integral Inequalities"*,Advances in dynamical Systems and Applications,3(2), pp. 291-296.`

[2] Xu, Y. C., and Xia, F. X., 2005, *"A Retarded Integral Inequality and its Applications"*, Qufu Shifan Daxue Xuebao Ziran Kexue Ban, 31(1), pp. 15-18.

[3] Lipouan, O., 2003, *"A Retarded Integral Inequality and its Applications"*, J. Math. Anal. Appl., 285, pp. 436-443.

[4] Pachpatte,B.G., 1998, *"Inequalities For Differential and Integral Equations"*, Academic Press, Newyork,.