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Upper Limits to the Complex Growth Rate in Couple-Stress Fluid in the Presence of Rotation

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Abstract

The thermal instability of a couple-stress fluid acted upon by uniform vertical rotation and heated from below is investigated. Following the linearized stability theory and normal mode analysis, the paper through mathematical analysis of the governing equations of couple-stress fluid convection with a uniform vertical rotation for the case of rigid boundaries shows that the complex growth rate σ of oscillatory perturbations, neutral or unstable for all wave numbers, must lie inside a semi-circle

$$|\sigma|^2 = T_A - \left(\pi^4\right),$$

in the right half of a complex σ-plane, which prescribes the upper limits to the complex growth rate of arbitrary oscillatory motions of growing amplitude in a rotatory couple-stress fluid heated from below.

Keywords: Thermal convection; Couple-Stress Fluid; Rotation; Complex growth rate; Taylor number.

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Introduction

Right from the conceptualizations of turbulence, instability of fluid flows is being regarded at its root. The thermal instability of a fluid layer with maintained adverse temperature gradient by heating the underside plays an important role in Geophysics, interiors of the Earth, Oceanography and Atmospheric Physics etc. A detailed account of the theoretical and experimental study of the onset of Bénard Convection in Newtonian fluids, under varying assumptions of hydrodynamics and hydromagnetics, has been given by Chandrasekhar [3]. The use of Boussinesq approximation has been made throughout, which states that the density changes are disregarded in all other terms in the equation of motion except the external force term. Sharma et al [7] has considered the effect of suspended particles on the onset of Bénard convection in hydromagnetics. The fluid has been considered to be Newtonian in all above studies. With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. Stokes [11] proposed and postulated the theory of couple-stress fluid. One of the applications of couple-stress fluid is its use to the study of the mechanism of lubrication of synovial joints, which has become the object of scientific research. According to the theory of Stokes [11], couple-stresses are found to appear in noticeable magnitude in fluids having very large molecules. Since the long chain hylauronic acid molecules are found as additives in synovial fluid, Walicki and Walicka [12] modeled synovial fluid as couple-stress fluid in human joints. An electrically conducting couple-stress fluid heated from below in porous medium in the presence of uniform horizontal magnetic field has been studied by Sharma and Sharma [10]. Sharma and Thakur [8] have studied the thermal convection in couple-stress fluid in porous medium in hydromagnetics. Sharma and Sharma [9] and Kumar and Kumar [4] have studied the effect of dust particles, magnetic field and rotation on couple-stress fluid heated from below and for the case of stationary convection, found that dust particles have destabilizing effect on the system, where as the rotation is found to have stabilizing effect on the system, however couple-stress and magnetic field are found to have both stabilizing and destabilizing effects under certain conditions.

Banerjee et al [2] gave a new scheme for combining the governing equations of thermohaline convection which is shown to lead to bounds for the complex growth rate of the arbitrary oscillatory perturbations, neutral or unstable for all combinations of dynamically rigid or free boundaries. Keeping in mind the importance of non-Newtonian fluids, the present paper is an attempt to prescribe the upper limits to the complex growth rate of arbitrary oscillatory motions of growing amplitude, in a layer of incompressible couple-stress fluid heated from below in the presence of uniform vertical rotation opposite to force field of gravity, when the bounding surfaces of infinite horizontal extension, at the top and bottom of the fluid are rigid.

Formulation of the Problem and Perturbation Equations

Considered an infinite, horizontal, incompressible couple-stress fluid layer, of thickness d, heated from below so that, the temperature and density at the bottom surface z = 0 are T_0, ρ_0 respectively and at the upper surface z = d are T_d, ρ_d and that a uniform adverse temperature gradient $\beta\left(=\left|\frac{dT}{dz}\right|\right)$ is maintained. The fluid is acted upon by a uniform vertical rotation $\vec{\Omega}(0,0,\Omega)$. Let ρ, p, T and $\vec{q}(u,v,w)$ denote respectively the density, pressure, temperature and velocity of the fluid. Then the momentum balance, mass balance equations of the couple-stress fluid (Stokes[11]; Chandrasekhar[3] and Scanlon and Segel[5]) are

$$\frac{\partial \vec{q}}{\partial t}+\left(\vec{q}.\nabla\right)\vec{q}=-\frac{1}{\rho_0}\nabla p+\vec{g}\left(1+\frac{\delta\rho}{\rho_0}\right)+\left(\nu-\frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2\vec{q}+2\left(\vec{q}\times\vec{\Omega}\right), \tag{1}$$

$$\nabla.\vec{q}=0, \tag{2}$$

The equation of state

$$\rho=\rho_0\left[1-\alpha\left(T-T_0\right)\right], \tag{3}$$

Where the suffix zero refer to the values at the reference level z = 0. Here $\vec{g}(0,0,-g)$ is acceleration due to gravity.

Let c_v, c_{pt} denote the heat capacity of the fluid at constant volume and the heat capacity of the particles. Assuming that the particles and the fluid are in thermal equilibrium, the equation of heat conduction gives

$$\rho_0 c_v\left(\frac{\partial}{\partial t}+\vec{q}.\nabla\right)T=\vec{q}\,\nabla^2 T,$$

Or

$$\frac{\partial T}{\partial t}+\left(\vec{q}.\nabla\right)T=\kappa\nabla^2 T, \tag{4}$$

The kinematic viscosity ν, couple-stress viscosity μ', thermal diffusivity κ, and coefficient of thermal expansion α are all assumed to be constants.

The basic motionless solution is

$$\vec{q}=(0,0,0),\ T=T_0-\beta z,\ \vec{\Omega}=(0,0,\Omega) \text{ and } \rho=\rho_0(1+\alpha\beta z). \tag{5}$$

Assume small perturbations around the basic solution and let $\delta\rho$, δp, θ and $\vec{q}(u,v,w)$ denote respectively the perturbations in density, pressure p, temperature T and couple-stress fluid velocity (0,0,0) . The change in density $\delta\rho$ caused mainly by the perturbation θ in temperature is given by

$$\delta\rho=-\alpha\rho_0\theta. \tag{6}$$

Then the linearized perturbation equations of the couple-stress fluid becomes

$$\frac{\partial \vec{q}}{\partial t} = -\frac{1}{\rho_0}\nabla \delta p - \vec{g}\,\alpha\theta + \left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2 \vec{q} + 2\left(\vec{q}\times\vec{\Omega}\right), \tag{7}$$

$$\nabla.\vec{q} = 0, \tag{8}$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa\nabla^2\theta, \tag{9}$$

Where

$$\kappa = \frac{q}{\rho_0 c_v}.$$

Within the framework of Boussinesq approximation, equations (7) and (8), give

$$\left[\frac{\partial}{\partial t}\nabla^2 w - g\alpha\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + 2\Omega\frac{\partial \varsigma}{\partial z}\right] = \left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\nabla^4 w, \tag{10}$$

$$\left[\frac{\partial \varsigma}{\partial t} - 2\Omega\frac{\partial w}{\partial z}\right] = \left(\nu - \frac{\mu'}{\rho_0}\nabla^2\right)\nabla^2\varsigma, \tag{11}$$

Together with (9), where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\varsigma = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ denote the z-component of vorticity.

Normal Mode Analysis

Analyzing the disturbances into normal modes, we assume that the Perturbation quantities are of the form

$$[w,\theta,\varsigma] = [W(z),\Theta(z),Z(z)]\ \mathrm{Exp}\left(ik_x x + ik_y y + nt\right) \tag{12}$$

Where k_x, k_y are the wave numbers along the x and y-directions respectively $k = \left(k_x^{\,2} + k_y^{\,2}\right)^{\frac{1}{2}}$, is the resultant wave number and n is the growth rate which is, in general, a complex constant.

Using (12), equations (9), (10) and (11), on using (8), in non-dimensional form, become

$$\left(D^2 - a^2\right)\left[\sigma + F\left(D^2 - a^2\right)^2 - \left(D^2 - a^2\right)\right]W = -\frac{g\alpha d^2 a^2\Theta}{\nu} - \sqrt{T_A}\,dDZ, \tag{13}$$

$$\left[\left\{1 - F\left(D^2 - a^2\right)\right\}\left(D^2 - a^2\right) - \sigma\right]Z = -\frac{\sqrt{T_A}}{d} DW \,, \tag{14}$$

$$\left(D^2 - a^2 - p_1\sigma\right)\Theta = -\frac{\beta d^2}{\kappa} W \,, \tag{15}$$

where $a = kd, \sigma = \frac{nd^2}{\nu}, p_1 = \frac{\nu}{\kappa}, F = \frac{\mu'}{\rho_0 d^2 \nu}, T_A = \frac{2\Omega^2 d^4}{\nu^2}$ $D = \frac{d}{dz}$ and $D_\oplus = dD$ and dropping $(\oplus)$ for convenience. Here $p_1 = \frac{\nu}{\kappa}$, is the thermal prandtl number, F is the couple-stress parameter and T_A is the Taylor number.

Substituting $W = W_\oplus$, $\Theta = \frac{\beta d^2}{\kappa}\Theta_\oplus$ and $Z = \frac{\sqrt{T_A}}{d} Z_\oplus$ in equations (13), (14) and (15) and dropping $(\oplus)$ for convenience, in non-dimensional form becomes,

$$\left(D^2 - a^2\right)\left[\sigma + F\left(D^2 - a^2\right)^2 - \left(D^2 - a^2\right)\right]W = -Ra^2\Theta - T_A DZ \,, \tag{16}$$

$$\left[\left\{1 - F\left(D^2 - a^2\right)\right\}\left(D^2 - a^2\right) - \sigma\right]Z = -DW \,, \tag{17}$$

$$\left(D^2 - a^2 - p_1\sigma\right)\Theta = -W \,, \tag{18}$$

Where $R = \frac{g\alpha\beta d^4}{\kappa\nu}$, is the thermal Rayleigh number.

Since both the boundaries rigid and are maintained at constant temperature, the perturbations in the temperature are zero at the boundaries. The appropriate boundary conditions with respect to which equations (16), (17) and (18) must be solved are

W = DW = 0, $\Theta = 0$ and Z = 0 at z = 0 and z = 1. (19)

Equations (16)-(18), along with boundary conditions (19), pose an eigenvalue problem for σ and we wish to Characterize σ_i when $\sigma_r \geq 0$.

We prove the following theorem:

Theorem: If R$\rangle$ 0, F$\rangle$ 0, $T_A \rangle 0$, $\sigma_r \geq 0$ and $\sigma_i \neq 0$ then the necessary condition for the existence of non-trivial solution (W, Θ, Z) of equations (16), (17) and (18) together with boundary conditions (19) is that

$$|\sigma|^2 \langle T_A - \left(\pi^4\right) .$$

Proof: Multiplying equation (16) by W^* (the complex conjugate of W) throughout and integrating the resulting equation over the vertical range of z, we get

$$\sigma\int_0^1 W^*\left(D^2 - a^2\right)W dz + F\int_0^1 W^*\left(D^2 - a^2\right)^3 W dz - \int_0^1 W^*\left(D^2 - a^2\right)^2 W dz = -Ra^2\int_0^1 W^*\Theta dz - T_A\int_0^1 W^* DZ d, \tag{20}$$

Taking complex conjugate on both sides of equation (18), we get

$$\left(D^2 - a^2 - p_1\sigma^*\right)\Theta^* = -W^*, \tag{21}$$

Therefore, using (21), we get

$$\int_0^1 W^*\Theta dz = -\int_0^1 \Theta\left(D^2 - a^2 - p_1\sigma^*\right)\Theta^* dz, \tag{22}$$

Also taking complex conjugate on both sides of equation (17), we get

$$\left[\left\{1 - F\left(D^2 - a^2\right)\right\}\left(D^2 - a^2\right) - \sigma^*\right]Z^* = -DW^*, \tag{23}$$

Therefore, using (23), we get

$$\int_0^1 W^* DZdz = -\int_0^1 DW^* Zdz = \int_0^1 Z\left\{\left(D^2 - a^2\right) - F\left(D^2 - a^2\right)^2 - \sigma^*\right\}Z^* dz, \tag{24}$$

Substituting (22) and (24) in the right hand side of equation (20), we get

$$\sigma\int_0^1 W^*\left(D^2 - a^2\right)Wdz + F\int_0^1 W^*\left(D^2 - a^2\right)^3 Wdz - \int_0^1 W^*\left(D^2 - a^2\right)^2 Wdz$$
$$= Ra^2\int_0^1 \Theta\left(D^2 - a^2 - p_1\sigma^*\right)\Theta^* dz - T_A\int_0^1 Z\left\{\left(D^2 - a^2\right) - F\left(D^2 - a^2\right)^2 - \sigma^*\right\}Z^* dz, \tag{25}$$

Integrating the terms on both sides of equation (25) for an appropriate number of times by making use of the appropriate boundary conditions (19), along with (17), we get

$$\sigma\int_0^1 \left\{|DW|^2 + a^2|W|^2\right\}dz + F\int_0^1 \left\{|D^3W|^2 + 3a^2|D^2W|^2 + 3a^4|DW|^2 + a^6|W|^2\right\}dz$$

$$+\int_0^1 \left\{|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2\right\}dz = Ra^2\int_0^1 \left\{|D\Theta|^2 + a^2|\Theta|^2 + p_1\sigma^*|\Theta|^2\right\}dz - T_A\int_0^1 \left\{|DZ|^2 + a^2|Z|^2\right\}dz$$
$$- T_A F\int_0^1 \left\{|D^2Z|^2 + 2a^2|DZ|^2 + a^4|Z|^2\right\}dz - T_A\sigma^*\int_0^1 |Z|^2 dz, \tag{26}$$

And equating imaginary parts on both sides of equation (26), and cancelling $\sigma_i (\neq 0)$ throughout from imaginary part, we get

$$\int_0^1 \left\{|DW|^2 + a^2|W|^2\right\}dz + Ra^2 p_1\int_0^1 |\Theta|^2 dz = T_A\int_0^1 |Z|^2 dz, \tag{27}$$

We first note that since W and Z satisfy $W(0) = 0 = W(1)$ and $Z(0) = 0 = Z(1)$ in addition to satisfying to governing equations and hence we have from the Rayleigh-Ritz inequality [6]

$$\int_0^1 |DW|^2 dz \geq \pi^2 \int_0^1 |W|^2 dz \,, \tag{28}$$

and

$$\int_0^1 |DZ|^2 dz \geq \pi^2 \int_0^1 |Z|^2 dz \,, \tag{29}$$

Further, for $W(0) = 0 = W(1)$ and $Z(0) = 0 = Z(1)$, Banerjee et al. [1] have shown that

$$\int_0^1 \left|D^2 W\right|^2 dz \geq \pi^2 \int_0^1 |DW|^2 dz \ \text{ and } \int_0^1 \left|D^2 Z\right|^2 dz \geq \pi^2 \int_0^1 |DZ|^2 dz \,, \tag{30}$$

Further, multiplying equation (17) and its complex conjugate (23), and integrating by parts each term on both sides of the resulting equation for an appropriate number of times and making use of boundary condition on Z namely $Z(0) = 0 = Z(1)$ along with (17), we get

$$\int_0^1 \left\{ \left|D^2 Z\right|^2 + 2a^2 |DZ|^2 + a^4 |Z|^2 \right\} dz + F \int_0^1 \left\{ \left|D^4 Z\right|^2 + 4a^2 \left|D^3 Z\right|^2 + 6a^4 \int_0^1 \left|D^2 Z\right|^2 + 4a^6 \int_0^1 |DZ|^2 + a^8 |Z|^2 \right\} dz$$

$$+ 2F \int_0^1 \left\{ \left|D^3 Z\right|^2 + 3a^2 \left|D^2 Z\right|^2 + 3a^4 \int_0^1 |DZ|^2 + a^6 |Z|^2 \right\} dz + 2\sigma_r \int_0^1 \left\{ |DZ|^2 + a^2 |Z|^2 \right\} dz$$

$$+ 2\sigma_r F \int_0^1 \left\{ \left|D^4 Z\right|^2 + 2a^2 \left|D^2 Z\right|^2 + a^4 |Z|^2 \right\} dz + |\sigma|^2 \int_0^1 |Z|^2 dz = \int_0^1 |DW|^2 dz \,, \tag{31}$$

Now F $\rangle$ 0 and $\sigma_r \geq 0$, therefore the equation (31) gives,

$$\int_0^1 \left|D^2 Z\right|^2 dz + |\sigma|^2 \int_0^1 |Z|^2 dz \langle \int_0^1 |DW|^2 dz \,, \tag{32}$$

And on utilizing the inequalities (29) and (30), inequality (32) gives

$$\int_0^1 |Z|^2 dz \langle \frac{1}{\left(\pi^4 + |\sigma|^2\right)} \int_0^1 |DW|^2 dz \,, \tag{33}$$

Now R $\rangle$ 0 and $T_A \rangle$ 0, utilizing the inequalities (33), the equation (27) gives,

$$\left[1 - \frac{T_A}{\left(\pi^4 + |\sigma|^2\right)}\right]\int_0^1 |DW|^2 dz + a^2 \int_0^1 |W|^2 dz + Ra^2 p_1 \int_0^1 |\Theta|^2 dz \,\langle\, 0, \tag{34}$$

and therefore , we must have

$$|\sigma|^2 \langle T_A - \left(\pi^4\right). \tag{35}$$

Hence, if

$\sigma_r \geq 0$ and $\sigma_i \neq 0$, then $|\sigma|^2 \langle T_A - \left(\pi^4\right)$.

And this completes the proof of the theorem.

Conclusions

The essential content of the theorem, from the point of view of linear stability theory is that for the configuration of couple-stress fluid of infinite horizontal extension heated form below, having top and bottom bounding surfaces rigid, in the presence of uniform vertical rotation parallel to the force field of gravity, the complex growth rate of an arbitrary oscillatory motions of growing amplitude, lies inside a semi-circle in the right half of the $\sigma_r \sigma_i$ - plane whose centre is at the origin and radius is $\sqrt{T_A - \left(\pi^4\right)}$.

Further, it follows from inequality (35) that a sufficient condition for the validity of the 'principle of exchange of stabilities' in rotatory couple-stress fluid convection is that the non-dimensional number $\frac{T_A}{\left(\pi^4\right)} \leq 1$. It is therefore clear that the existence of oscillatory motions of growing amplitude in this problem depends crucially upon the magnitude of the non-dimensional number $\frac{T_A}{\left(\pi^4\right)}$, in the sense so long as $0 \langle \frac{T_A}{\left(\pi^4\right)} \leq 1$, no such motions are possible.

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