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Analytic Expressions of Roche-Harmonics for Stellar Models Distorted By Differential Rotation and Tidal Distortion

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Abstract

Kopal obtained the explicit expressions for Roche-harmonics associated with the stellar model of such star, which is rotating on its axis of rotation as a solid body. However, it may be possible that many stars are not rotating under solid body rotation, they have differential rotations. Keeping this in view, Singh and Gupta have obtained the explicit expressions for Roche-harmonics of star, which is distorted by differential rotation using the law of rotation of the form $\omega = b_1 + b_2 s^2$. To draw more conclusions, we extend the analysis for Roche-harmonics associated with the Roche-model of Star distorted by differential rotation using the law of the form $\omega = b_1 + b_2 s^2 + b_3 s^4$. We have also derived the analytic expressions for Roche-coordinates and Roche-harmonics of non-rotating but tidally distorted star. This can be more useful to understand the nature, equilibrium structure, oscillations and stability of the star.

Introduction

Kopal [3] introduced a family of new auxiliary functions, which he called Roche-harmonics, generated by the solution of Laplace equation $\nabla^2\phi = 0$. The solution of Laplace equation has been derived in terms of curvilinear coordinates, which are associated with the Roche-equipotential. Roche [8] discussed some more theoretical aspects of Roche-harmonics. In fact, as was pointed by Kopal (*op.cit.*), spherical harmonics may be regarded as limiting case of the more general Roche-harmonics, and the latter may be appropriate for the study of more problems arising in double stars Astronomy. Observations show that, there are certain variable stars which are rotating on their axes as well as about each other (cf. Tassoul [12], Slettback [11]). To study the problem of rotationally and tidally distorted stars, Kopal [2, 3] obtained the explicit expressions for Roche-coordinates and Roche-harmonics for stellar models distorted by solid body rotation and as well as distorted by tidal forces. Mohan and Singh [6, 7] and thereafter Singh [10] extended the analysis of Roche-coordinates taking into account the effects of differential rotation. The law of differential rotation has been assumed to be the form $\omega = b_1 + b_2 s^2$. Singh and Gupta [9] used the Roche-coordinates obtained by Mohan and Singh [7], to extend the analysis and have derived the new harmonics for stars. In this paper we present the analysis of Roche-harmonics for stellar models distorted by differential rotation, assuming the generalized law of differential rotation of the form $\omega = b_1 + b_2 s^2 + b_3 s^4$, where ω is the angular velocity of an element distant s from the axis of rotation and b_1, b_2 and b_3 are constants.

The contents of this paper are as follows: The explicit expressions of Roche-coordinates, Metric-coefficients and Roche-harmonics have been obtained, in section 2, for Roche-equipotential surface of a star distorted by differential rotation using the law $\omega = b_1 + b_2 s^2 + b_3 s^4$. In section 3, the explicit expressions of Roche-coordinates, Metric-coefficients and Roche-harmonics have been obtained for Roche-equipotential surface of a star distorted by tidal forces. Concluding remarks are reported in section 4.

Roche-Coordinates for Roche-Model Distorted by Differential Rotation Obeying the Law of the Form $\omega = b_1 + b_2 s^2 + b_3 s^4$

In the case of actual stars, the greater part of their mass is concentrated very near to the centre. Therefore, their structure comes much closer to the Roche-model (by Roche-model we mean a model in which the whole mass of the star is supposed to be concentrated at the centre and this point mass is surrounded by an evanescent envelope in which density is assumed to vary inversely as some positive power of the distance from the centre). On the basis of some extensive numerical investigations, Chandrasekhar [1] has shown that for stars whose central density bears to the mean density a ratio of 100 or more (as is likely to be true for many of the main sequence stars let alone be the red giants), the rotating Roche-model of a rotating configuration represents the actual form of the equipotential surfaces of a rotating star within an error of less than one percent. It is, therefore, reasonable to assume that equipotential surfaces of the type of relation (3) reasonably approximate the equipotential surfaces of the most of the rotating stars in close binary system.

In our present study, let M and M' be the total masses of the primary and secondary components of a binary system, which are assumed to be gaseous spheres. Let D be the mutual separation between centers of these two masses. Further suppose that the position of two components of this binary system is referred to a rectangular system of cartesian-coordinates having the origin at the centre of gravity of mass M, the x-axis along the line joining the centers of the two components and z-axis is perpendicular to the plane of the orbit of two components. In this system of coordinates, the centre of gravity G of the system may be written as $G\left(\frac{M'D}{M+M'}, 0{,}0\right)$.

For the Roche-model of mass M, rotating according to the law $\omega = b_1 + b_2 s^2 + b_3 s^4$, the equation of hydrostatic equilibrium may be written in the form

$$d\psi = dV + \frac{1}{2}\omega^2 d(s^2), \tag{1}$$

where ψ denotes the potential at a point P at distance r from the centre of star and $V = GM/r$ is the gravitational potential, ω is the angular velocity of an element distance s of the star from the axis of rotation.

On using the law of differential rotation and relation (1), we can express our scheme as

$$d\psi = dV + \frac{1}{2}[(b_1^2 + 2b_1 b_2 s^2 + (2b_1 b_3 + b_2^2)s^4 + 2b_2 b_3 s^6 + b_3^2 s^8) d(s^2) \tag{2}$$

On integration and simplification equation (2) gives

$$\psi = \frac{GM}{r} + \frac{1}{2}\left[b_1^2(x^2+y^2) + b_1 b_2 (x^2+y^2)^2 + \frac{1}{3}(2b_1 b_3 + b_2^2)(x^2+y^2)^3 + \frac{1}{2}b_2 b_3 (x^2+y^2)^4 \right.$$
$$\left. + \frac{1}{5} b_3^2 (x^2+y^2)^5 \right]. \tag{3}$$

In terms of spherical polar coordinates $x = r\cos\phi \sin\vartheta = r\lambda$, $y = r\sin\phi\sin\vartheta = r\mu$ and $z = r\cos\vartheta = r\nu$, expression (3) becomes

$$\xi = \frac{1}{r} + \frac{1}{2}r^2(1-\nu^2)\left[b_1^2(1-\nu^2) + b_1 b_2 r^2(1-\nu^2) + \frac{1}{3}(2b_1 b_3 + b_2^2) r^4 (1-\nu^2)^2 + \frac{1}{2} b_2 b_3 r^6 (1-\nu^2)^3 \right.$$
$$\left. + \frac{1}{5} b_3^2 r^8 (1-\nu^2)^4 \right], \tag{4}$$

where $\xi = D\psi/GM$ is a non-dimensional parameter denoting potential, ω is non-dimensional angular velocity in the unit of GM/D^3 . The surfaces generated by setting $\xi =$constant in (4) are referred as the Roche-equipotentials. Now if we take $r_0 = \xi^{-1}$ as our scheme of first approximation to the distance of the equipotential surfaces from the centre then it can be shown by taking second approximation as $r_1 = r_0 + \Delta' r = r_0(1 + \Delta' r/r_0)$ and proceeding similarly for higher order approximations (r_2 , r_3 ,. ..), we have

$$r = r_0\left[1+\frac{1}{12}r_0^3(1-\nu_0^2)\{6b_1^2 + 6b_1b_2r_0^2(1-\nu_0^2) + 2(2b_1b_3 + b_2^2)r_0^4(1-\nu_0^2)^2 + 3b_2b_3r_0^6(1-\nu_0^2)^3 + b_3^2r_0^8(1-\nu_0^2)^4\} + ...\right], \tag{5}$$

where $\nu_0 = \cos\zeta$.

In the system of Roche-coordinates (ξ,η,ξ), the ξ-coordinate is defined by Roche-equipotential surface of the form as (4) while coordinates η and ζ are defined by requirement that they are orthogonal of ξ as well as to each other. The triple orthogonal system of Roche-coordinates given by $\xi_x\eta_x + \xi_y\eta_y + \xi_z\eta_z = 0$, $\xi_x\zeta_x + \xi_y\zeta_y + \xi_z\zeta_z = 0$ and $\eta_x\zeta_x + \eta_y\zeta_y + \eta_z\zeta_z = 0$, the second and third coordinates can be obtained as

$$\eta = \lambda\big/\sqrt{1-\nu^2} \tag{6}$$

and

$$\cos\varsigma = \nu\left[1-\frac{1}{3465}r^3(1-\nu^2)\left\{1155b_1^2 + 1386b_1b_2r^2(1-\nu^2) + 495(2b_1b_3 + b_2^2)r^4(1-\nu^2)^2 + 770b_2b_3r^6(1-\nu^2)^3 + 315b_3^2r^8(1-\nu^2)^4\right\} + ...\right], \tag{7}$$

where ξ and η are exact but ζ is correct up to second order terms in ω.

The metric coefficients h_1, h_2 and h_3 are defined by $h_1 = \left(\xi_x^2 + \xi_y^2 + \xi_z^2\right)^{-1/2}$, $h_2 = \left(\eta_x^2 + \eta_y^2 + \eta_z^2\right)^{-1/2}$ and $h_3 = \left(\zeta_x^2 + \zeta_y^2 + \zeta_z^2\right)^{-1/2}$, where suffixes denote partial differentiation with respect to x, y and z. By these relations we have found

$$h_1(\xi,\zeta) = r_0^2 + 2b_1^2r_0^5(1-\nu^2) + 3b_1b_2r_0^7(1-\nu_0^2)^2 + \frac{4}{3}(2b_1b_3 + b_2^2)r_0^9(1-\nu_0^2)^3 + \frac{5}{2}b_2b_3r_0^{11}(1-\nu_0^2)^4 + ...$$

$$h_2(\xi,\zeta) = r_0\sin\zeta\left[1 - r_0^3\left\{\frac{1}{3}b_1^2 + \frac{2}{5}b_1b_2r_0^2(1-\nu_0^2) + \frac{1}{7}(2b_1b_3 + b_2^2)r_0^4(1-\nu_0^2)^2 + \frac{2}{9}b_2b_3r_0^6(1-\nu_0^2)^3 + \frac{1}{11}b_3^2r_0^8(1-\nu_0^2)^4\right\} + r_0^3\sin^2\zeta\left\{\frac{5}{6}b_1^2 + \frac{9}{10}b_1b_2r_0^2(1-\nu_0^2) + \frac{13}{42}(2b_1b_3 + b_2^2)r_0^4(1-\nu_0^2)^2 + \frac{17}{36}b_2b_3r_0^6(1-\nu_0^2)^3 + \frac{23}{132}b_3^2r_0^8(1-\nu_0^2)^4\right\}....\right]$$

$$h_3(\xi,\zeta)=r_0\left[1-r_0^3\left\{\frac{2}{3}b_1^2+\frac{8}{5}b_1b_2r_0^2(1-\nu_0^2)+\frac{6}{7}(2b_1b_3+b_2^2)r_0^4(1-\nu_0^2)^2+\frac{16}{9}b_2b_3r_0^6(1-\nu_0^2)^3\right.\right.$$
$$\left.+\frac{10}{11}b_3^2r_0^8(1-\nu_0^2)^4\right\}+r_0^3\sin^2\zeta\left\{\frac{3}{2}b_1^2+\frac{5}{2}b_1b_2r_0^2(1-\nu_0^2)+\frac{7}{6}(2b_1b_3+b_2^2)r_0^4(1-\nu_0^2)^2+\right.$$
$$\left.\left.+\frac{9}{4}b_2b_3r_0^6(1-\nu_0^2)^3+\frac{13}{12}b_3^2r_0^8(1-\nu_0^2)^4\right\}....\right] \tag{8}$$

On using the metric coefficients as reported in equation (8), the Laplacian operator $\nabla^2\equiv\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi}\left(\frac{h_2h_3}{h_1}.\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_1h_3}{h_2}.\frac{\partial}{\partial\eta}\right)+\frac{\partial}{\partial\varsigma}\left(\frac{h_1h_2}{h_2}.\frac{\partial}{\partial\varsigma}\right)\right]$ becomes:

$$\nabla^2\equiv\left\{\left[1-3b_1^2r_0^3(1-\nu_0^2)-4b_1b_2r_0^5(1-\nu_0^2)^2-5(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^3+\frac{20}{9}b_2b_3r_0^9(1-\nu_0^2)^4\right.\right.$$
$$\left.-3b_1^2r_0^3-10b_1b_2r_0^5(1-\nu_0^2)-7(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^2-20b_2b_3r_0^9(1-\nu_0^2)^3\right]\frac{\partial}{\partial r_0}\left(r_0^2\frac{\partial}{\partial r_0}\right)$$
$$-\left[2-\frac{1}{3}b_1^2r_0^3+\frac{8}{3}b_1^2r_0^3(1-\nu_0^2)-\frac{22}{5}b_1b_2r_0^5(1-\nu_0^2)^2-\frac{38}{11}(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)\right.$$
$$\left.+\frac{44}{5}b_1b_2r_0^5(1-\nu_0^2)+\frac{44}{7}(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^2+\frac{232}{7}b_2b_3r_0^9\right]\left(\nu_0\frac{\partial}{\partial\nu_0}\right)+\left[1-\frac{1}{6}b_1^2r_0^3\right.$$
$$-\frac{3}{2}b_1^2r_0^3(1-\nu_0^2)-5b_1b_2r_0^5(1-\nu_0^2)^2-\frac{7}{3}(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^3+\frac{16}{5}b_1b_2r_0^5(1-\nu_0^2)$$
$$\left.\left.+\frac{12}{7}(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^2+\frac{29}{7}b_2b_3r_0^9\right](1-\nu_0^2)\frac{\partial^2}{\partial\nu_0^2}\right\} \tag{9}$$

The Laplace equation therefore becomes

$$\left\{\left[1-3b_1^2r_0^3(1-\nu_0^2)-4b_1b_2r_0^5(1-\nu_0^2)^2-5(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^3+\frac{20}{9}b_2b_3r_0^9(1-\nu_0^2)^4\right.\right.$$
$$\left.-3b_1^2r_0^3-10b_1b_2r_0^5(1-\nu_0^2)-7(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^2-20b_2b_3r_0^9(1-\nu_0^2)^3\right]\frac{\partial}{\partial r_0}\left(r_0^2\frac{\partial\phi}{\partial r_0}\right)$$
$$-\left[2-\frac{1}{3}b_1^2r_0^3+\frac{8}{3}b_1^2r_0^3(1-\nu_0^2)-\frac{22}{5}b_1b_2r_0^5(1-\nu_0^2)^2-\frac{38}{11}(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)\right.$$
$$\left.+\frac{44}{5}b_1b_2r_0^5(1-\nu_0^2)+\frac{44}{7}(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^2+\frac{232}{7}b_2b_3r_0^9\right]\left(\nu_0\frac{\partial\phi}{\partial\nu_0}\right)+\left[1-\frac{1}{6}b_1^2r_0^3\right.$$

$$-\frac{3}{2}b_1^2r_0^3(1-\nu_0^2)-5b_1b_2r_0^5(1-\nu_0^2)^2-\frac{7}{3}(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^3+\frac{16}{5}b_1b_2r_0^5(1-\nu_0^2)$$

$$\left.\left.+\frac{12}{7}(2b_1b_3+b_2^2)r_0^7(1-\nu_0^2)^2+\frac{29}{7}b_2b_3r_0^9\right](1-\nu_0^2)\frac{\partial^2\phi}{\partial\nu_0^2}\right\}=0 \tag{10}$$

Let the solution of this equation be obtained in a series form as

$$\phi=\sum_j a_j r_0^j R_j\ \text{, and } R_j=P_j(\nu_0)+\frac{w^2}{2}r_0^3X_2^{(j)}(\nu_0)+\frac{w^2}{4}r_0^6X_4^{(j)}(\nu_0)+\dots, \tag{11}$$

where
$$w=b_1^2+2b\,b_2r_0^2(1-\nu_0^2)+(2b_1b_3+b_2^2)r_0^4(1-\nu_0^2)^2+2b_2b_3r_0^6(1-\nu_0^2)^3+b_3^2r_0^8(1-\nu_0^2)^4+\dots.$$

On using (11) in (10), it is shown that the functions $x_2^{(j)}(\nu_0)$ and $x_4^{(j)}(\nu_0)$ assume the form because of the influence of the terms pertaining to $b_2^2, b_1b_2\ 2b_1b_3+b_2^2$ and b_3^2 as given by

$$x_2^{(2)}(\nu_0)=-0.26293\nu_0^4+0.30172\nu_0^2-2.71551, \tag{12}$$

$$x_2^{(3)}(\nu_0)=-3.125\nu_0^5+1.21428\nu_0^3-3.625\nu_0, \tag{13}$$

$$x_2^{(4)}(\nu_0)=-16.5625\nu_0^6+31.71726\nu_0^4-14.46428\nu_0^2+19.6428, \tag{14}$$

$$x_2^{(5)}(\nu_0)=-17.4444\nu_0^7+13.5196\nu_0^5-2.61284\nu_0^3+3.83246\ \nu_0, \tag{15}$$

$$x_2^{(6)}(\nu_0)=-9.30416\nu_0^8+40.425\nu_0^6-62.2777\nu_0^4+22.75\nu_0^2-1.1325, \tag{16}$$

$$x_4^{(2)}(\nu_0)=-0.01947\nu_0^6+0.29114\nu_0^4-0.3115\nu_0^2+2.71551, \tag{17}$$

$$x_4^{(3)}(\nu_0)=-0.51562\nu_0^7+11.4922\nu_0^5-23.2889\ \nu_0^3+2.64553\ \nu_0, \tag{18}$$ etc.

Explicit Expressions for Roche-Coordinates with the Terms upto Second-Order of Smallness in Tidal Effect

For a star distorted by solid body rotation as well as tidal distortion, Kopal [2] has shown that

$$\frac{D\psi}{GM}-\frac{M'^2}{2M(M+M')}=\frac{1}{r}+q\left\{\frac{1}{\sqrt{(1-2\lambda r+r^2)}}-\lambda r\right\}+nr^2(1-\nu^2), \tag{19}$$

where, $q=M'/M$ and $n=(q+1)/2=\omega^2/2$ are non-dimensional parameters.

By expanding the radical $\sqrt{(1-2\lambda r+r^2)}$ in terms of Legendre's polynomial

$P_j(\lambda)$ with $n = 0$ (for a non-rotating but tidally distorted star), relation (19) gives:

$$\xi = \frac{1}{r} + q\{1 + \textstyle\sum_{j=2}^{4} r^j\, P_j(\lambda_0)\} = \text{constant}\,, \qquad (20)$$

where, $\xi = \frac{D\psi}{GM} - \frac{M'^2}{2M(M+M')}$ is non-dimensional potential.

Now if we take $r_0 = \frac{1}{\xi - q}$ as our first approximation to the distance of equipotential surface from the centre of the star of mass M, it can be shown that

$$r = r_0\left\{1 + q\sum_{j=2}^{4} r_0^j\, P_j(\lambda_0)\right\} \text{ and } (1-\lambda^2) = (1-\lambda_0^2)\left\{1 + 2q\sum_{j=2}^{4}\frac{r_0^{j+1}}{j+1}P_j'(\lambda_0)\right\},$$

where $\lambda_0 = \cos\eta$.

In present tidal case of triply orthogonal system Roche-coordinate η now becomes

$$\left.\begin{aligned} \eta &= cos^{-1}\lambda - \frac{q}{(1-\lambda^2)}\textstyle\sum_{j=2}^{9}\frac{r_0^{j+1}}{j+1}P_j(\lambda) \\ &\text{and } \zeta = \cos^{-1}\frac{\nu}{1-\lambda^2} \end{aligned}\right\} \qquad (21)$$

respectively. Whereas the expression ζ is in closed analytical form, expression for η contains only terms up to second order in smallness in q.

Subsequent to the analysis of section 2, the explicit expressions for metric-coefficients h_1, h_2 and h_3 up to second-order terms in q now become

$$\begin{aligned} h_1(\xi,\eta) = r_0^2 \Bigg[1 + q\sum_{j=2}^{9}(j+2)r^{j+1}\,P_j(\lambda_0) \\ + q^2\sum_{j=2}^{4}\sum_{k=2}^{4} r_0^{j+k+2}\,\{(3 + j(j+k+5))P_j(\lambda_0)P_k(\lambda_0) \\ -\frac{(j+5)(1-\lambda_0^2)}{2(j+1)}P_j'(\lambda_0)P_k'(\lambda_0)\Big\} + \cdots \Bigg], \end{aligned}$$

$$\begin{aligned} h_2(\xi,\eta) = r_0 \Bigg[1 + q\sum_{j=2}^{9}\{(j+1)^2 P_j(\lambda_0) - \lambda_0 P_j'(\lambda_0)\} \\ +q^2\sum_{j=2}^{4}\sum_{k=2}^{4}\frac{(j+2)r_0^{j+k+2}}{(j+1)}\{j(j+1)P_j(\lambda_0)P_k(\lambda_0) - \lambda_0 P_j'(\lambda_0)P_k(\lambda_0)\} \\ +q^2\sum_{j=2}^{4}\sum_{k=2}^{4} r_0^{j+k+2}\left\{(j+1)P_j(\lambda_0)P_k(\lambda_0) - \frac{(j+3)(1-\lambda_0^2)}{2(j+1)}P_j'(\lambda_0)P_k'(\lambda_0)\right\} \\ -\frac{P_j'(\lambda_0)P_k'(\lambda_0)}{2(j+1)(k+1)} + \frac{jP_j(\lambda_0)}{(k+1)}\{k(k+1)P_k(\lambda_0)\} - \lambda_0 P_k'(\lambda_0) + \cdots \Bigg], \end{aligned}$$

$$h_3(\xi,\eta) = r_0\sqrt{1-\lambda_0^2}\Bigg[1 + q\sum_{j=2}^{9}\frac{r^{j+k+1}}{j+1}\{(j+1)P_j(\lambda_0) + \lambda_0 P_j'(\lambda_0)\}$$
$$+\lambda_0 q^2 \sum_{j=2}^{4}\sum_{k=2}^{4}\frac{r_0^{j+k+2}}{(j+1)}\{(j+1)P_j(\lambda_0) + \lambda_0 P_j'(\lambda_0)\}P_j(\lambda_0)P_k'(\lambda_0)$$
$$+q^2\sum_{j=2}^{4}\sum_{k=2}^{4} r_0^{j+k+2}\left[\{j(j+1) + (j+1)(k+1)\}P_j(\lambda_0)P_k'(\lambda_0)\right]$$
$$+q^2\sum_{j=2}^{4}\sum_{k=2}^{4} r_0^{j+k+2}\Big\{(j+1)P_j(\lambda_0)P_k(\lambda_0) + \Big(\frac{1}{2(j+1)(k+1)}$$
$$+\frac{(1-\lambda_0^2)}{(j+1)}\Big)P_j(\lambda_0)P_k'(\lambda_0)\Big\} + \cdots\Bigg]. \qquad (22)$$

On using the metric coefficients as reported by the relations (22), the Laplacian operator (∇^2) becomes

$$\nabla^2 \equiv \Big[16qr_0\lambda_0^2 + \frac{15}{4}qr_0^2(\lambda_0^3 - 1) + q\,r_0^3(42\lambda_0^4 - 24\lambda_0^2)$$
$$-\frac{1}{16}qr_0^4(24885\lambda_0^5 - 9870\lambda_0^3 + 245\lambda_0)$$
$$-\frac{1}{16}q^2r_0^4(1080\lambda_0^5 + 5136\lambda_0^4 - 6993\lambda_0^3 + 240\lambda_0^2 + 881\lambda_0)$$
$$+\cdots\Big]\frac{\partial}{\partial\lambda_0}\{(1-\lambda_0^2)\}\frac{\partial}{\partial\lambda_0}$$
$$+\Big[\frac{1}{2} + \frac{1}{2}qr_0^3(50\lambda_0^2 - 51) + \frac{1}{8}qr_0^4(401\lambda_0^3 - 75\lambda_0 - 1)$$
$$+\frac{1}{8}qr_0^5(1197\lambda_0^4 - 1038\lambda_0^2 - 231)$$
$$+\frac{1}{32}qr_0^6(65710\lambda_0^5 - 39480\lambda_0^3 - 14773\lambda_0 - 1792)$$
$$+\frac{1}{16}q^2r_0^6(1296\lambda_0^5 - 5016\lambda_0^4 - 10101\lambda_0^3 - 540\lambda_0^2 + 5286\lambda_0 + 1698) + \cdots\Big]\frac{\partial^2}{\partial{\lambda_0}^2}.$$

Therefore in this case the Laplace equation ($\nabla^2\phi = 0$) takes the explicit form as

$$\nabla^2\phi \equiv \Big[16qr_0\lambda_0^2 + \frac{15}{4}qr_0^2(\lambda_0^3 - 1) + q\,r_0^3(42\lambda_0^4 - 24\lambda_0^2)$$
$$-\frac{1}{16}qr_0^4(24885\lambda_0^5 - 9870\lambda_0^3 + 245\lambda_0)$$
$$-\frac{1}{16}q^2r_0^4(1080\lambda_0^5 + 5136\lambda_0^4 - 6993\lambda_0^3 + 240\lambda_0^2 + 881\lambda_0)$$
$$+\cdots\Big]\frac{\partial}{\partial\lambda_0}\{(1-\lambda_0^2)\}\frac{\partial\phi}{\partial\lambda_0}$$

$$+\left[\frac{1}{2}+\frac{1}{2}qr_0^3(50\lambda_0^2-51)+\frac{1}{8}qr_0^4(401\lambda_0^3-75\lambda_0-1)\right.$$
$$+\frac{1}{8}qr_0^5(1197\lambda_0^4-1038\lambda_0^2-231)$$
$$+\frac{1}{32}qr_0^6(65710\lambda_0^5-39480\lambda_0^3-14773\lambda_0-1792)$$
$$\left.+\frac{1}{16}q^2r_0^6(1296\lambda_0^5-5016\lambda_0^4-10101\lambda_0^3-540\lambda_0^2+5286\lambda_0+1698)+\cdots\right]\frac{\partial^2\phi}{\partial{\lambda_0}^2}=0\,. \quad (23)$$

If we assume $\phi=\sum_j a_j r_0^j R_j$ as a series solution of equation (23), where

$$R_j=P_j(\lambda_0)+q\sum_{i=2}^4 r_0^{i+1}Y_i^{(j)}(\lambda_0)+q^2\sum_{i=2}^4\sum_{j=2}^4 r_0^{i+1}r_0^{j+1}Y_i^{(j)}(\lambda_0)Y_j^{(i)}(\lambda_0)+\cdots, \quad (24)$$

On neglecting higher order terms than q^2, it can be shown that the functions $Y_2^{(j)}(\lambda_0)$, $Y_3^{(j)}(\lambda_0)$ and $Y_4^{(j)}(\lambda_0)$ assume the form

$$Y_2^{(2)}(\lambda_0)=7.5324\lambda_0^4-6.0763\lambda_0^2+0.4978\,, \quad (25)$$
$$Y_2^{(3)}(\lambda_0)=18.7532\lambda_0^5-19.5876\lambda_0^3+3.8789\lambda_0\,, \quad (26)$$
$$Y_2^{(4)}(\lambda_0)=43.75\lambda_0^6-56.2598\lambda_0^4+16.426\lambda_0^2-0.7625\,, \quad (27)$$
$$Y_3^{(2)}(\lambda_0)=13.125\lambda_0^5-13.7686\lambda_0^3+9.2618\lambda_0\,, \quad (28)$$
$$Y_3^{(3)}(\lambda_0)=32.812\lambda_0^6-42.1876\lambda_0^4+13.76426\lambda_0^2-0.5625\,, \quad (29)$$
$$Y_3^{(4)}(\lambda_0)=41.265\lambda_0^7-58.2761\lambda_0^5+8.2753\lambda_0^3-3.7899\lambda_0\,, \quad (30)$$
$$Y_4^{(2)}(\lambda_0)=23.625\lambda_0^6-28.625\lambda_0^4+8.3761\lambda_0^2-0.3752\,, \quad (31)$$
$$Y_4^{(3)}(\lambda_0)=52.2832\lambda_0^7-18.7148\lambda_0^5+26.869\lambda_0^3-0.9876\lambda_0\,. \quad (32)$$

Expression (24) with (25) to (32) constitutes the explicit form of Roche-harmonics associated with the Roche-equipotential surface (20) of differentially rotating Roche-model of star up-to second-order of approximation in tidal distortion.

In the present paper, the explicit expressions of Roche-coordinates ξ,η and ζ are given by relations (4), (6) and (7) respectively for differentially rotating Roche-model of star. The expressions for ξ and η are found to be exact whereas the expression for ζ coordinate is correct upto second-order terms only. Like the explicit expressions of Roche-coordinates obtained by Kopal [5] for solid body rotation and by Mohan and Singh [6], for differential rotation, the explicit expressions of Roche-coordinates for Roche-model of star rotating differentially according to the law of rotation $\omega=b_1+b_2s^2+b_3s^4$), are useful to study the problems of oscillations and stability of such stars.

Concluding Remarks

In the scheme of our approximation, if we neglect terms pertaining to b_2 , b_3 and retaining b_1 terms then $x_2^{(j)}$ is same as obtained by Kopal [3], and if, we neglect only b_3 term, then $x_2^{(j)}$ is same as obtained by Singh and Gupta [9]. In the present analysis, we could only present the results which are because of the influence of the terms pertaining to $b_2^2, b_1b_2, 2b_1b_2 + b_2^2, b_2b_3$ and b_3^2 because our scheme of approximation for angular velocity ω is only up to second order of smallness, and in this, higher order terms have been neglected. The consideration of higher order terms in ω and the combined effect of differential rotation, tidal distortion, Coriolis force and magnetic perturbations, should lead to the appropriate formulation of this problem which we intended to investigate in our subsequent study.

It may be pointed out that although we have studied the problems of Roche-coordinates associated with the equipotential surface by assuming the Roche-model of the star, the present method of Roche-coordinates can also be used when some more realistic structure is assumed for the interior of the model. We can still either approximate the distorted equipotentials of such stars by Roche-model or use their more realistic form by using the system of Clairaut's-coordinates (cf. Kopal [4, 5]), when the law of differential rotation may be assumed of the form $\omega = b_1 + b_2s^2 + b_3s^4$.

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