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Asymptotic behavior of Solutions of Generalized Nonlinear Difference Equations of Second Order

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Abstract

In this paper, the authors discuss the asymptotic behavior of solutions of the generalized nonlinear difference equation

$$\Delta_\ell(p(k)\Delta_\ell u(k)) + f(k)F(u(k)) = g(k), \qquad (1)$$

$k \in [a, \infty)$, where, the functions p, f, F and g are defined in their domain of definition and ℓ is a positive real. Further, $uF(u) > 0$ for $u \neq 0, p(k) > 0$ for all $k \in [a, \infty)$ for some $a \in [0, \infty)$ and for all $0 \le j < \ell, R_{a+j,k} \to \infty$, where

$$R_{t+j,k} = \sum_{r=0}^{\frac{k-\ell-t-j}{\ell}} \frac{1}{p(t+j+r\ell)}, \; t \in [a, \infty) \text{ and } k \in \mathbb{N}_\ell(t+j+\ell).$$

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1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k),\ k \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Eventhough many authors ([1], [10]-[12]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k),\ k \in \mathbb{R},\ \ell \in \mathbb{N}(1), \tag{2}$$

no significant progress is noticed on this line. But recently, when E. Thandapani, M.M.S. Manuel and G.B.A. Xavier considered the definition of Δ as given in (2), the theory of difference equations flourished in a different direction (see [4]-[5]). For convenience, the operator Δ defined by (2) was labelled as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory (See [4],[7]-[9]) were obtained. By extending the study related to sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike were analysed for the solutions of difference equations involving Δ_ℓ. The results obtained using Δ_ℓ can be found in ([4-9]).

In [3], John R. Graef worked on Oscillation, nonoscillation, and growth of solutions of nonlinear functional differential equations of arbitrary order and Blazej Szmanda [2] obtained the discrete analogous of [3]. In [2] the author considered $\ell = 1$ and $k \in \mathbb{N}(a)$ for an integer a but, in this paper the theory is extended for all real $k \in [a, \infty)$ and for any real ℓ and oscillation, nonoscillation and growth of solutions of the generalized nonlinear difference equation (1) is discussed.

Throughout this paper we make use of the following notations.

(a) $\mathbb{N} = \{0, 1, 2, 3, \ldots\},\ \mathbb{N}(a) = \{a, a+1, a+2, \ldots\}$,

(b) $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \ldots\}$.

(c) $\lceil x \rceil$ upper integer part of x.

2. Preliminaries

Definition 2.1. [4] Let $u(k),\ k \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the generalized difference operator Δ_ℓ is defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k). \tag{3}$$

Similarly, the generalized difference operator of the r^{th} kind is defined as

$$\Delta_\ell^r u(k) = \underbrace{\Delta_\ell(\Delta_\ell(\ldots(\Delta_\ell u(k))\ldots))}_{r\ times}. \tag{4}$$

Definition 2.2. **[4]** Let $u(k)$, $k \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} is defined as follows.

$$\text{If } \Delta_\ell v(k) = u(k), \ \text{ then } v(k) = \Delta_\ell^{-1} u(k) + c_j, \tag{5}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - \left[\frac{k}{\ell}\right]\ell$.

In general $\Delta_\ell^{-n} u(k) = \Delta_\ell^{-1}(\Delta_\ell^{-(n-1)} u(k))$ for $n \in \mathbb{N}(2)$.

Lemma 2.3. **[4]** If the real valued function $u(k)$ is defined for all $k \in [a, \infty)$, then

$$\Delta_\ell^{-1} u(k) = \sum_{r=1}^{\left[\frac{k-a}{\ell}\right]} u(k - r\ell) + c_j, \tag{6}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - a - [\frac{k-a}{\ell}]\ell$.

Theorem 2.4. If $\Delta_\ell v(k) = u(k)$ for $k \in [k_2, \infty)$ and $j = k - k_2 - [\frac{k-k_2}{\ell}]\ell$, then

$$v(k) - v(k_2 + j) = \sum_{r=0}^{\left[\frac{k-k_2-j-\ell}{\ell}\right]} u(k_2 + j + r\ell).$$

Proof. The proof follows by Definition 2.2, Lemma 2.4 and $c_j = v(k_2 + j)$. ■

Definition 2.5. The solution $u(k)$ of (1) is called oscillatory if for any $k_1 \in [a, \infty)$ there exists a $k_2 \in \mathbb{N}_\ell(k_1)$ such that $u(k_2)u(k_2+\ell) \leq 0$. The difference equation itself is called oscillatory if all its solutions are oscillatory. If the solution $u(k)$ is not oscillatory, then it is said to be nonoscillatory (i.e. $u(k)u(k + \ell) > 0$ for all $k \in [k_1, \infty)$).

3. Main Results

In this section, we present conditions for the oscillation and nonoscillation of equation (1).

Theorem 3.1. Consider the generalized difference equation

$$\Delta_\ell(p(k)\Delta_\ell u(k)) + f(k)F(u(k)) = 0 \tag{7}$$

and assume that in addition to the given hypotheses on the functions p, f and $F, |F(u)|$ is bounded away from zero if $|u|$ is bounded away from zero, $f(k) \geq 0$ for all $k \in [a, \infty)$ and $\sum_{r=0}^{\infty} f(k_1 + j + r\ell) = \infty$, then equation (7) is oscillatory.

Proof. Let $u(k)$ be a nonoscillatory solution of (1) and suppose that $u(k) > 0$ eventually. From the given hypothesis, there exists a positive constant c such that $F(u(k)) \geq c$ for all $k \in [k_2, \infty)$.

On the other hand from (1), we have

$$\Delta_\ell(p(k)\Delta_\ell u(k)) + cf(k) \leq 0, k \in [k_1, \infty) \tag{8}$$

and hence by Definition 2.2 and Theorem 2.4 we obtain

$$p(k)\Delta_\ell u(k) \leq -c \sum_{r=0}^{\frac{k-j-k_2-\ell}{\ell}} f(k_2 + j + r\ell) \to -\infty \text{ as } k \to \infty.$$

We then have $\Delta_\ell u(k) \leq -1/p(k)$. Again by Definition 2.2 and Theorem 2.4,

$$u(k) \leq -\sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} \frac{1}{p(a+j+r\ell)} \to -\infty \text{ as } k \to \infty,$$

$k \in [k_2, \infty)$, where $j = k - k_2 - \left\lceil \frac{k-k_2}{\ell} \right\rceil \ell$. This leads to a contradiction to our assumption that $u(k) > 0$ eventually. The case $u(k) < 0$ eventually can be treated similarly. ■

Example 3.2. For the generalized difference equation $\Delta_\ell(k\Delta_\ell u(k)) - u(k)(9k+6\ell) = 0$, and for $p(k) = k$, $f = (9k + 6\ell)$, $F(u(k)) = -u(k)$, the conditions of Theorem 3.1 hold and hence the generalized difference equation is oscillatory. Infact $u(k) = (-2)^{\left\lceil \frac{k}{\ell} \right\rceil}$ is one such solution.

Theorem 3.3. Suppose that the following conditions hold

(i) $f(k) \geq b > 0$ for all $k \in [a, \infty)$

(ii) $|F(u)|$ is bounded away from zero if $|u|$ is bounded away from zero

(iii) the function $G(k) = \sum_{r=0}^{\frac{k-\ell-a-j}{\ell}} g(a+j+r\ell)$ is bounded on $[a, \infty)$.

Then, for every nonoscillatory solution $u(k)$ of (1), $\lim_{k\to\infty} u(k) = 0$.

Proof. In system form, equation (1) is equivalent to

$$\Delta_\ell u(k) = (v(k) + G(k))/p(k) \tag{9}$$

$$\Delta_\ell v(k) = -f(k)F(u(k)). \tag{10}$$

If $u(k)$ is a nonoscillatory solution of (1), then we can assume that $u(k) > 0$ eventually (the case $u(k) < 0$ can be similarly treated). First, we shall show that $\liminf_{k\to\infty} u(k) = 0$. If not, then there exist $k_1 \geq a$ and a positive constant c_1 such that $F(u(k)) \geq c_1$ for all $k \in [k_1, \infty)$. From (10) it follows that

$$\begin{aligned} v(k+\ell) - v(k_1) &= -\sum_{r=0}^{\frac{k-k_1-j}{\ell}} f(k_1+j+r\ell)F(u(k_1+j+r\ell)) \\ &\leq -c_1 \sum_{r=0}^{\frac{k-k_1-j}{\ell}} f(k_1+j+r\ell) \to -\infty \text{ as } k \to \infty. \end{aligned}$$

We then have $\Delta_\ell u(k) = (v(k) + G(k))/p(k) \leq -1/p(k)$ for all $k \in [k_2, \infty)$, for some $k_2 \geq k_1$. This implies that $u(k) \leq u(k_2) - \sum_{r=0}^{\frac{k-\ell-k_2-j}{\ell}} 1/p(k_2+j+r\ell) \to -\infty$ as $k \to \infty$. But, this contradicts the fact that $u(k)$ is eventually positive. From the above argument, we also have

$$\sum_{r=0}^{\infty} f(k_1+j+r\ell)F(u(k_1+j+r\ell)) < \infty. \tag{11}$$

If $\limsup_{k\to\infty} u(k) = \gamma > 0$, then there exists a sequence $\{k_t\} \subseteq [0, \infty)$, such that $u(k_t) \to \gamma$ as $t \to \infty$. Hence, there is $t(0)(k_{t(0)} \geq a)$ such that $u(k_t) \geq \gamma/2$ and $F(u(k_t)) \geq c_2$ for all $t \geq t(0)$, where c_2 is a positive constant. But, then we have

$$\begin{aligned} &\sum_{r=0}^{\frac{k_t-k_{t(0)}-j}{\ell}} f(k_{t(0)}+j+r\ell)F(u(k_{t(0)}+j+r\ell)) \\ &\geq \sum_{r=0}^{\frac{t-k_{t(0)}-j}{\ell}} f(k_{t(0)+j+r\ell})F(u(k_{t(0)+j+r\ell})) \\ &\geq bc_1(t-t(0)+\ell) \to \infty \end{aligned}$$

as $t \to \infty$, so that $\sum_{r=0}^{\infty} f(k_1+j+r\ell)F(u(k_1+j+r\ell)) = \infty$ which contradicts (11). This completes the proof. ■

Example 3.4. For the generalized difference equation

$$\Delta_\ell(k\Delta_\ell u(k)) + \frac{k^2(2k^2\ell + 5k\ell^2 + \ell^3)u(k)}{(k+\ell)^2(k+2\ell)^2} = \frac{2\ell(k-\ell)}{k(k+\ell)^2},$$

and for $p(k) = k$, $f = 2k^2\ell + 5k\ell^2 + \ell^3$, $F(u(k)) = \dfrac{k^2u(k)}{(k+\ell)^2(k+2\ell)^2}$, the conditions of Theorem 3.3 hold and hence all nonoscillatory solutions of the generalized difference equation satisfies $\lim_{k\to\infty} u(k) = 0$. Infact one such solution is $u(k) = \dfrac{1}{k^2}$.

Theorem 3.5. In addition to the condition (ii) let

(iv) $f(k) > 0$ for all $k \in [a, \infty)$, and $\sum_{r=0}^{\infty} f(k_1 + j + r\ell) = \infty$ and

(v) $\lim_{k\to\infty} g(k)/f(k) = 0$.

Then, for every nonoscillatory solution $u(k)$ of (1), $\lim\inf_{k\to\infty} |u(k)| = 0$.

Proof. Let $u(k)$ be a nonoscillatory solution of (1), say, $u(k) > 0$ for all $k \in [k_1, \infty)$, where $k_1 \geq a$. Then, $u(k)$ is also a nonoscillatory solution of $\Delta_\ell(p(k)\Delta_\ell u(k)) + [f(k) - g(k)/F(u(k))]F(u(k)) = 0, k \in [k_1, \infty)$. Suppose that $\lim\inf_{k\to\infty} u(k) > 0$, then by the hypotheses, there exists a positive constant c such that $F(u(k)) \geq c$ for all $k \in [k_1, \infty)$. Thus, by (v) there exists a $k_2 \geq k_1$ such that $\dfrac{g(k)}{(f(k)F(u(k)))} < \dfrac{1}{2}$ for all $k \in [k_2, \infty)$. This implies that

$$f(k) - \frac{g(k)}{F(u(k))} = f(k)\big[1 - \frac{g(k)}{(f(k)F(u(k)))}\big] \geq \frac{1}{2}f(k), k \in [k_2, \infty).$$

So, from (iv) we get

$$\sum_{r=0}^{\infty}\big[f(k_1 + j + r\ell) - \frac{g(k_1 + j + r\ell)}{F(u(k_1 + j + r\ell))}\big] = \infty.$$

But, then by Theorem 3.1, $u(k)$ must be oscillatory. This contradiction completes the proof. ■

Example 3.6. For the generalized difference equation

$$\Delta_\ell(k\Delta_\ell u(k)) + \frac{(k+\ell)^2(2k^2\ell + 7k\ell^2 + 5\ell^3)u(k)}{(k+3\ell)^2(k+2\ell)^2} = \frac{k\ell(2k-3\ell)}{(k+2\ell)^2(k+\ell)^2},$$

and for $p(k) = k$, $f = 2k^2\ell + 7k\ell^2 + 5\ell^3$, $F(u(k)) = \dfrac{(k+\ell)^2u(k)}{(k+3\ell)^2(k+2\ell)^2}$, the conditions of Theorem 3.3 hold and hence all nonoscillatory solutions of the generalized difference equation satisfies $\lim_{k\to\infty} |u(k)| = 0$. Infact one such solution is $u(k) = \dfrac{1}{(k+\ell)^2}$.

Theorem 3.7. In addition to the condition (iv) let

(vi) $F(u)$ is continuous at $u = 0$ and

(vii) $$\liminf_{k\to\infty} \frac{\sum_{r=0}^{\frac{k-t-j}{\ell}} g(t+j+r\ell)}{\sum_{r=0}^{\frac{k-t-j}{\ell}} f(t+j+r\ell)} \geq c > 0 \text{ for every } t \in [a,\infty).$$

Then, no solution of (1) approaches zero.

Proof. Let $u(k)$ be a solution of (1) which approaches zero. Then, by the hypotheses on the function F, there exists a $k_1 \geq a$ such that $F(u(k)) < c/4$ for all $k \in [k_1, \infty)$. Hence, from the equation (1) we have

$$\begin{aligned} &p(k+\ell)\Delta_\ell u(k+\ell) - p(k_1+j)\Delta_\ell u(k_1+j) \\ &\geq -\frac{c}{4} \sum_{r=0}^{\frac{k-k_1-j}{\ell}} f(k_1+j+r\ell) + \sum_{r=0}^{\frac{k-k_1-j}{\ell}} g(k_1+j+r\ell), \end{aligned}$$

which by (vii) yields

$$\frac{p(k+\ell)\Delta_\ell u(k+\ell)}{\sum_{r=0}^{\frac{k-k_1-j}{\ell}} f(k_1+j+r\ell)} - \frac{p(k_1+j)\Delta_\ell u(k_1)+j}{\sum_{r=0}^{\frac{k-k_1-j}{\ell}} f(k_1+j+r\ell)}$$

$$\geq -\frac{c}{4} + \frac{\sum_{r=0}^{\frac{k-k_1-j}{\ell}} g(k_1+j+r\ell)}{\sum_{r=0}^{\frac{k-k_1-j}{\ell}} f(k_1+j+r\ell)} \geq -\frac{c}{4} + \frac{c}{2} = \frac{c}{4} > 0,$$

for all large k. Now, because of (iv) the above inequality implies that $p(k)\Delta_\ell u(k) \to \infty$ as $k \to \infty$, which in turn leads to the contradictive conclusion that $u(k) \to \infty$ as $k \to \infty$. ■

Example 3.8. For the generalized difference equation

$$\Delta_\ell(k\Delta_\ell u(k)) - ku(k) = \ell 2^{\lceil \frac{k}{\ell} \rceil},$$

and for $F(u(k)) = -ku(k)$, $f = 1$, $g = \ell 2^{\lceil \frac{k}{\ell} \rceil}$, the conditions of Theorem 3.7 hold and hence no solution of the generalized difference equation approaches zero. $u(k) = 2^{\lceil \frac{k}{\ell} \rceil}$ is one such solution.

Remark 3.9. If we replace conditions (iv) and (vii) by (iv)$'$ $f(k) < 0$ for all $k \in [a, \infty)$, and $\sum_{r=0}^{\infty} f(t + j + r\ell) = -\infty$

(v)$'$ $\limsup_{k\to\infty} \sum_{r=0}^{\frac{k-t-j}{\ell}} g(t + j + r\ell) / \sum_{r=0}^{\frac{k-t-j}{\ell}} f(t + j + r\ell) \leq c < 0$ for every $t \in [a, \infty)$,

then, the assertion of Theorem 3.7 holds.

Theorem 3.10. Suppose that the following conditions hold

(viii) $F(u)$ is locally bounded in $[0, \infty)$ and

(ix) $\sum_{r=0}^{\infty} |f(t + j + r\ell)| < \infty, \sum_{r=0}^{\infty} g(t + j + r\ell) = \infty.$

Then, every solution of (1) is unbounded.

Proof. Let $u(k)$ be a bounded solution of (1), i.e. $|u(k)| < M$, where M is a positive constant. Then, by (viii) there exist constants L_1 and L_2 such that $L_1 \leq F(u(k)) \leq L_2$. But then, from (1) and (ix), we obtain

$$\begin{aligned} &p(k + \ell)\Delta_\ell u(k + \ell) - p(a)\Delta_\ell u(a) \\ &\geq \sum_{r=0}^{\frac{k-a-j}{\ell}} g(a + j + r\ell) - L_2 \sum_{r=0}^{\frac{k-a-j}{\ell}} f^+(a + j + r\ell) - L_1 \sum_{r=0}^{\frac{k-a-j}{\ell}} f^-(a + j + r\ell) \end{aligned}$$

which tends to ∞, as $k \to \infty$. However, this leads to that $u(k) \to \infty$. This contradiction completes the proof. ∎

Example 3.11. For the generalized difference equation

$$\Delta_\ell(\frac{1}{k}\Delta_\ell u(k)) + \frac{u(k)(k + \ell) + \ell^3}{k(k + \ell)} = k,$$

and for $F(u(k)) = u(k)(k+\ell)+\ell^3$, $f = \dfrac{1}{k(k + \ell)}$, the conditions of Theorem 3.10 hold and hence all the solutions of the generalized difference equation are unbounded. Infact $u(k) = k^2$ is one such solution.

Remark 3.12. It is clear that Theorem 3.10 holds if we replace (ix) by $\sum_{r=0}^{\infty} g(t+j+r\ell) = -\infty$.

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