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A Mathematical Model of Power Law Fluid with an Application of Blood Flow through an Artery with Stenosis

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Abstract

The nature of blood flow changes from its usual state to a distributed flow condition due to the presence of a stenosis in the artery. Therefore the effect of size of stenosis on blood flow through an artery, where blood behave like a power law fluid in a uniform circular tube with an axially non-symmetric but radially symmetric stenosis. The governing equation for laminar, incompressible and non-Newtonian fluid (power-law fluid) flow subject to the boundary conditions is solved numerically. The analytical expressions for pressure drop and flux (flow rate) dimensionless resistance to flow and wall shear stress have been obtained. The numerical values extracted from these analytical expressions are presented graphically. Here in this investigation, shear stress is taken less than $20\sec^{-1}$ and the diameter of the tube is less than $0.2\,mm$, and observed that if the height of stenosis increased the pressure drop and flux also increased, the resistance to flow, wall shear stress decreases as stenosis size increases. This information of blood could be useful in the development of new diagnosis tools for many diseases.

Keywords: stenosis, flux, pressure drop, circular tube, non-Newtonian fluid.

1. Introduction

It has been suggested that the deposition of cholesterol on the arterial wall and proliferation of connective tissue may be responsible for the abnormal growth in the lumen of the artery. Many cardiovascular diseases such as due to the hemodynamics behavior of the blood flow is influenced by the presence of the arterial stenosis. If the stenosis is present in an artery, normal blood flow is disturbed. The intimal thickening of stenotic artery was understood as an early process in the beginning of atherosclerosis, while is the leading cause of death in many countries. There is considerable evidence that vascular fluid dynamics plays an important role in the development and progression of arterial stenosis, which is one of the most widespread diseases in human beings. The fluid mechanical study, specially of power law fluid of blood flow in artery bears some important aspects due to the engineering interest as well as the feasible medical applications. Stenosis is one of the most wide-spread arterial diseases. Effect of stenosis on the cardiovascular system has been determined by the studying the flow characteristics of blood in stenotic region in artery.

During the past few decay several studies were conducted in this direction of power law fluid and blood flow problems. Young (1979) described many problems in fluid mechanics of arterial stenosis, while Shukla *et al.* (1980) studied the effect of stenosis on non-Newtonian flow of blood in an artery. It has been suggested that the deposits of cholesterol on the arterial wall and proliferation of connective tissue may be responsible for the abnormal growth in the lumen of an artery. The actual causes of stenosis are not well known but its effect on the cardiovascular system can be understood by studying the blood flow in its vicinity. Ahmed and Giddens (1983) studied about the velocity measurements in steady flow through axisymmetric stenosis at moderate Reynolds number, while Back *et al.* (1986) realizing the fact that the pulsatile nature of the flow cannot be neglected, many theoretical analysis and experimental studies of the flow through stenosis have been performed. Haldar (1987) deal with the problem of oscillatory blood flow through a rigid tube with mild constriction under a simple harmonic pressure gradient and has examined the effect of stenosis on the flow field. It has been reported by Tu and Deville (1992) that the assumption of Newtonian behaviour of blood is acceptable for high shear rate flow. It has also been pointed out that in some diseased conditions (e.g. patients with severe myocardial infarction) cerebrovascular diseases and hypertension, blood exhibits remarkable non-Newtonian properties. Tu and Deville (1996) investigated the pulsatile flow of blood in stenosed arteries. Zendehbudi and Moayer (1999) have studied the comparison of physiological and simple pulsatile flows through stenosed arteries. It is true that the casson fluid model can be used for moderate shear rates $\gamma < 10s^{-1}$ in smaller diameter tubes whereas, the Herschel-Bulkley fluid model can be used at still lower shear rate of flow in very narrow arteries, where the yield stress is high reported by Ookawara and Ogowa (2000). Leuprecht and Perktold (2001) have studied the computer simulation of non-newtonian effects on blood flows in large arteries. Neofytou and Drikakis (2003) reported that a non-Newtonian flow instability in a channel with a sudden expansion. Chakravarty *et al.* (2004) presented a theoretical investigation to examine some of the significant characteristics of the two-layered non-Newtonian rheology of blood flowing through a tapered flexible artery in the presence of stenosis under a pulsatile pressure gradient. A mathematical model for blood flow in magnetic field is studied by Tzirtzilakis (2005) while Misra and Shit (2006, 2007) studied in two different situations on the blood flow through arterial stenosis by treating blood as a non-Newtonian (Herschel-Bulkley fluid) fluid model. It is generally well known that blood, being a suspension of red cells in plasma, behaves like a non-Newtonian fluid at low shear rates. Misra *et al.* (2008) also conducted a theoretical study for the effects of multiple stenosis, while also fluid flow analysis of blood flow through multistenosis arteries in the presence of magnetic field is investigated by Verma and Parihar (2010), where the effect of magnetic field and shape of stenosis on the flow rate is studied, Singh and Rathee (2010) studied the analytical solution of two-dimensional model of blood flow with variable viscosity through an indented artery due to LDL effect in the presence of magnetic field. A mathematical model of blood flow in porous vessel having double stenosis in the presence of an external magnetic field has been investigated by Sinha *et al.* (2011) while Shit and Roy (2012) put forwarded a mathematical analysis for the unsteady flow of blood through arteries having stenosis, in which blood was treated as a Newtonian viscous incompressible fluid. Recently Kumar and Diwakar (2012) worked on a biomagentic fluid dynamic model for the MHD couette flow between two infinite horizontal parallel porous plates and give that the main flow component decreases with the increase of hartmann number and the velocity decreases with the increases of the injection suction parameter, they also investigate the effects of hartmann number and suction parameter along with the velocity and temperature distribution. In same direction Kumar and Diwakar (2012) also discussed about a mathematical model for newtonian blood flow in the presence of applied magnetic field and found that results concerning the velocity and temperature field. Here the rate of heat transfer indicate that the presence of magnetic field appreciable influence the flow field, while the flow is appreciably influenced by the application of the magnetic field and in particularly by the strength and the magnetic field gradient.

Therefore on the basis of above information the power law fluid flow through a stenotis in artery is considered here.

2. Mathematical formulation

In this model we consider a circular cylinder arterial segment having axisymmetric and blood flow with this axially symmetric stenosed artery. Here the blood is assumed as a power law fluid and flow of blood is considered to be steady and laminar.

If w is the axial velocity, τ is the shear stress, τ_0 is the yield stress and μ is the viscosity of the blood then the constitutive equations may be expressed as:

$$e = f(\tau) = -\frac{dw}{dr} = \begin{cases} \left(\frac{\tau}{\mu}\right)^{1/n} & for \quad \tau \ge \tau_0 \\ 0 & for \quad \tau \le \tau_0 \end{cases} \qquad \ldots(i)$$

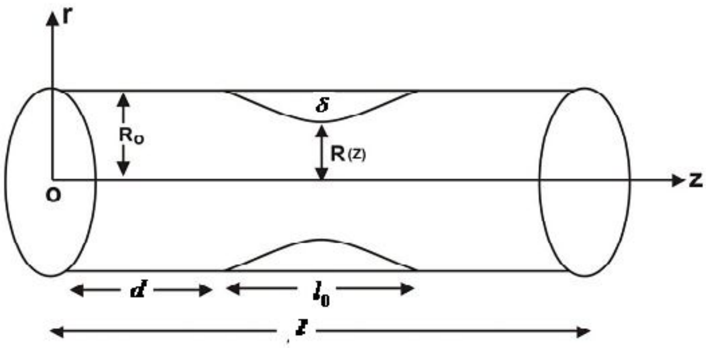

Figure 1: Schematic diagram of artery with stenosis.

Further if R_0 is the radius of artery, δ is the height of the stenosis and R is the radius of abnormal artery then the geometrical description of the figure 1 is given by the following equation:

$$\frac{R}{R_0} = \begin{cases} 1 - \frac{\delta}{2R_0}\left[1 + \cos\frac{2\pi}{l_0}\left(z - d - l_0/2\right)\right] & d \le z \le d + l_0 \\ 1 & Otherwise \end{cases} \qquad \ldots(ii)$$

Now if the velocity is parallel to the axis then it is a function of r only and the velocity is maximum on the axis and zero on the surface. So the non-zero component of srain rate is given by the Naveir-Stokes equation are:

$$e = -\left(\frac{dw}{dr}\right) \qquad \ldots(iii)$$

for the non-Newtonian fluid

$$\tau = f(e) = f\left(-\frac{dw}{dr}\right) \qquad \ldots(iv)$$

and the expression for τ is

$$\tau = \frac{1}{2}\mathrm{P}r \qquad \ldots(\mathrm{v})$$

from equation (iv) and (v) we have

$$-\frac{1}{2}\mathrm{P}r = f\left(-\frac{dw}{dr}\right) \qquad \ldots(\mathrm{vi})$$

for the power law fluid,

$$-\left(\frac{1}{2}\frac{\mathrm{P}r}{\mu}\right)^{1/n} = \left(\frac{dw}{dr}\right) \qquad \ldots(\mathrm{vii})$$

The boundry conditions are:

$$\left\{\begin{array}{lll} \partial w/\partial r = 0 & at & r = 0 \\ w = 0 & at & r = R_0 \end{array}\right\} \qquad \ldots(\mathrm{viii})$$

On integrating the equation (vii) and using the above boundry conditions (viii), we have:

$$w = \int_0^R \left(\frac{1}{2}\frac{\mathrm{P}r}{\mu}\right)^{1/n} dr$$

$$= \left(\frac{1}{2}\frac{\mathrm{P}}{\mu}\right)^{1/n} \int_0^R r^{1/n}\, dr$$

$$\text{or } w = \left(\frac{1}{2}\frac{\mathrm{P}}{\mu}\right)^{1/n} \frac{n}{n+1}\left[R^{1/n+1} - r^{1/n+1}\right] \qquad \ldots(\mathrm{ix})$$

Now the flux Q through the artery can be written in the following form:

$$Q = \int_0^R 2\pi\, r\, w\, dz$$

which may be expressed along with equation (ix):

$$Q = \frac{n\pi}{(3n+1)}\left(\frac{1}{2}\frac{\mathrm{P}}{\mu}\right)^{1/n}\left[R^{1/n+1}\right]^{1/n+3} \qquad \ldots(x)$$

or

$$\mathrm{P} = -\frac{dp}{dz} = 2\mu\left[\frac{(3n+1)Q}{n\pi}\right]^{n}\frac{1}{R^{3n+1}} \qquad \ldots(xi)$$

Integrating equation (xi) along the length of artery and using conditions:

$$p = p_1 \qquad at \qquad z = 0$$
$$p = p_2 \qquad at \qquad z = l$$

The pressure drop with no-slip condition

$$\Delta p = p_2 - p_1 = 2\mu\left[\frac{(3n+1)Q}{n\pi}\right]^{n}\int_0^l \frac{1}{R^{3n+1}}\,dz$$

$$\Delta p = 2\mu\left[\frac{(3n+1)Q}{n\pi}\right]^{n}\int_0^l \frac{1}{R^{3n+1}}\,dz \qquad \ldots(xii)$$

which can also be written as:

$$\Delta p = 2\mu\, l_0\left[\frac{(3n+1)Q}{n\pi}\right]^{n}\frac{1}{\pi R_0^{\;3n+1}}\int_0^l \frac{1}{(a-b\cos w)^{3n+1}}\,dw$$

$$\text{or } \Delta p = \frac{2\mu\, l_0}{\pi R_0^{\;3n+1}}\left[\frac{(3n+1)Q}{n\pi}\right]^{n}\int_0^l \frac{1}{(a-b\cos w)^{3n+1}}\,dw \qquad \ldots(xiii)$$

Now if there is no stenosis i.e. $\delta = 0$ and $f\left(\frac{\delta}{R_0}\right) = 1$ then the pressure drop across the stenosis length is given by:

$$(\Delta p)_{\delta=0} = \frac{2\mu\, l_0}{\pi R_0^{\;3n+1}}\left[\frac{(3n+1)Q}{n\pi}\right]^{n} \qquad \ldots(xiii)$$

Dividing the equation (xii) by equation (xiii) the we have,

$$\frac{\Delta p}{(\Delta p)_{\delta=0}}=\frac{1}{\pi}\int_0^{\pi}\frac{1}{(a-b\cos w)^{3n+1}}dw \qquad \ldots\text{(xiv)}$$

Now solve the equation (xiv) for the different value of n,

First we take $n=\frac{1}{3}$ then we have,

$$\frac{\Delta p}{(\Delta p)_{\delta=0}}=\left(1-\frac{\delta}{2R_0}\right)\left(1-\frac{\delta}{R_0}\right)^{-3/2} \qquad \ldots\text{(xv)}$$

Now for $n=\frac{2}{3}$,

$$\frac{\Delta p}{(\Delta p)_{\delta=0}}=\left(1-\frac{\delta}{R_0}+\frac{3}{8}\frac{\delta}{R_0}\right)\left(1-\frac{\delta}{R_0}\right)^{-5/2} \qquad \ldots\text{(xvi)}$$

Now for $n=1$

$$\frac{\Delta p}{(\Delta p)_{\delta=0}}=\left(1-\frac{\delta}{2R_0}\right)\left(1-\frac{\delta}{R_0}+\frac{3}{8}\frac{\delta^2}{R^2{}_0}\right)\left(1-\frac{\delta}{R_0}\right)^{-7/2} \qquad \ldots\text{(xvii)}$$

Similarly for the other different value of n, the definite integral (xiv) has to be evaluated numerically.

Again the resistance to flow or the resistive impedance is denoted by λ and is defined as:

$$\lambda=\frac{\Delta p}{Q}$$

From equations (x) and (xii), we expressed as:

$$\lambda_0=\frac{2\mu Q^{n-1}}{R_0^{3n+1}}\left[\frac{3n+1}{n\pi}\right]^n\int_0^l 1\Big/R_0^{3n+1}dz \qquad \ldots\text{(xvii)}$$

If there is no stenosis in artery then $\delta=0$, then the resistance to flow,

$$\lambda_N = \frac{2\mu l Q^{n-1}}{R_0^{3n+1}} \left[\frac{3n+1}{n\pi} \right]^n \qquad \ldots\text{(xix)}$$

The non-dimensional form of resistance to flow, denoted by $\overline{\lambda}$, is given as

$$\overline{\lambda} = \frac{\lambda_0}{\lambda_N}$$

$$\overline{\lambda} = 1 - \frac{l_0}{l} + \frac{1}{l} \int_d^{d+l_0} \frac{dz}{\left[R/R_0 \right]^{3n+1}} \qquad \ldots\text{(xx)}$$

The expression (xx) may be reduced to:

$$\overline{\lambda} = 1 - \frac{l_0}{l} + \frac{l_0}{2\pi l} \int_0^{2\pi l} \frac{dz}{\left[a + b\cos\theta \right]^{3n+1}}$$

Also the ratio of shearing stress on with and without stenosis can be written as;

$$\overline{\tau} = \frac{\tau_0}{\tau_N} = \left[\frac{R}{R_0} \right]^{3n}$$

where n is power law index.
Thus we get the graph for the different value of n as given below

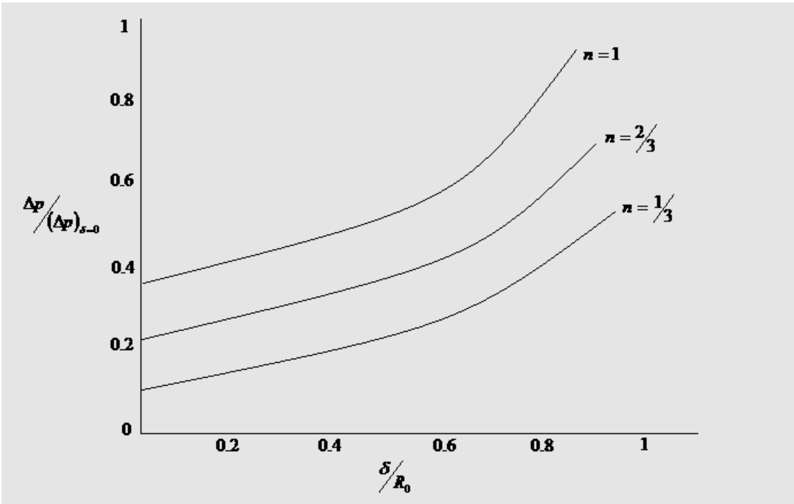

Figure 2: Pressure drop across stenosis size for power law fluid.

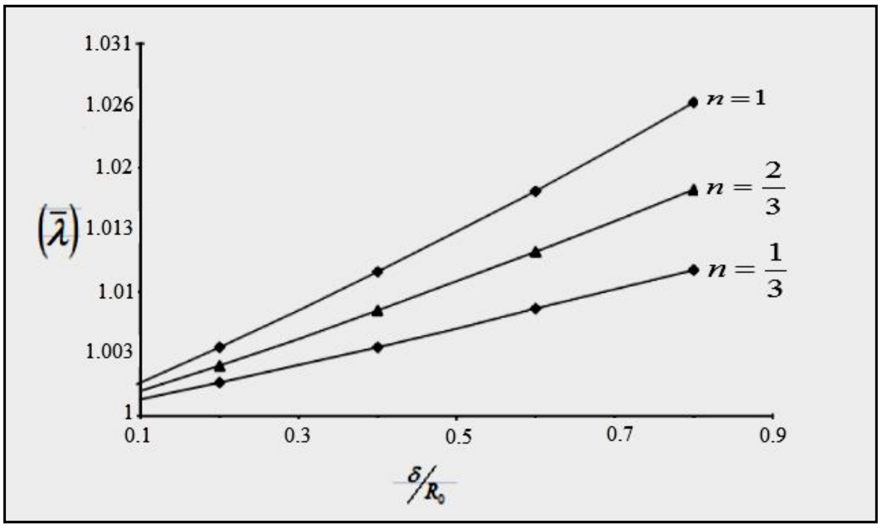

Figure 3: Resistence to flow with stenosis size for power law fluid.

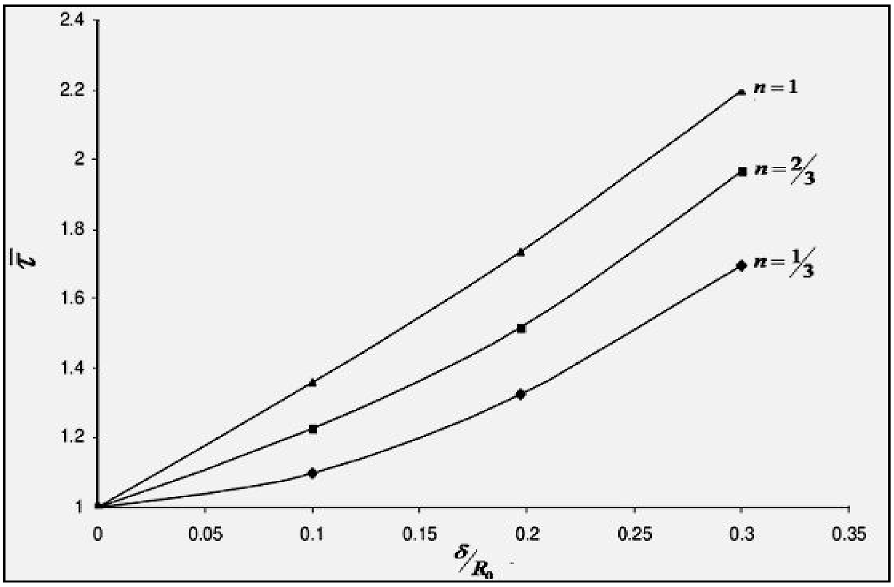

Figure 4: Wall shear stress with respect to stenosis size.

3. Numerical results and discussion

Blood flow through an artery mainly depends on the pressure gradient and resistance to flow. In this work, we have studied the effects of the stenosis in an artery by considering the blood as power-law fluid, and our result based on the mathematical analysis indicates that the pressure drop and flux varying markedly across the stenotic lesion. Here we see that if the size of stenosis increases the pressure drop and flux also increases show by figure 2, and figures 3 and 4 shows that the resistance to flow, wall shear stress decreases as stenosis size increases. We use the numerical tequnique to solve the analytical results of this model with considering the temperature $25.5^0 C$ and for this result the shear stress is taken less than $20\sec^{-1}$ and the diameter of the tube is less than $0.2\,mm$. It appears that the non-Newtonian behaviour of the blood is helpful in the functioning of diseased arterial circulation.

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