A Dynamical Model of the Spread of HIV/AIDS and Statistical Forecast for HIV/AIDS Population in India

Anulekha Tapadar¹, Koushik Ghosh² and Kripasindhu Chaudhuri³

¹Department of Mathematics Pailan College of Management and Technology Plot: 2-6, Sector-1, Phase I, Bengal Pailan Park Amgachia Road, Joka, Kolkata-700104, India E-mail: anulekha.ju@gmail.com ²Department of Mathematics, University Institute of Technology University of Burdwan, Burdwan-713 104, India E-mail: koushikg123@yahoo.co.uk ³UGC Emeritus Fellow, Department of Mathematics, Jadavpur University Kolkata-700 032, India E-mail: chaudhuriks@gmail.com

Abstract

Each year a large number of people all over the world die from HIV/AIDS. Although there are many complicating factors behind the spread of HIV, we still believe that relevant mathematical models can provide a good insight of the dynamics of the spread of it. If we can provide a satisfactory profile of this dynamics, it will certainly help government officials to make timely remedial actions. In the present work, we have proposed a mathematical model of the spread of HIV. We have made a search for equilibrium points for the system and discussed about their stabilities. On the basis of extensive analysis, relevant comments are made on mutual co-existence of the group infected by HIV and the group not infected by that. An effort is also made to evaluate the parameters involved in our model in the context of India. This practical study reveals a forecasting profile of the proportion and mass of HIV/AIDS infected people in India in coming years.

Keywords: HIV/AIDS, equilibrium points, stability, mutual co-existence, HIV/AIDS population in India.

Introduction

The Human Immunodeficiency Virus or HIV belongs to the family of Retroviruses, whose genetic material is RNA. In 1984, researchers discovered the primary causative viral agent, the human immunodeficiency virus type 1 (HIV-1). In 1986, a second type of HIV, called HIV-2 was discovered in West Africa, where it may have been present decades earlier. Both HIV-1 and HIV-2 have the same modes of transmission and are associated with similar opportunistic infections. In persons infected with HIV-2, immunodeficiency seems to develop more slowly. Compared with persons infected with HIV-1, those with HIV-2 are less infectious early in the course of infection [1].

HIV is transmitted by direct inoculation during intimate and unsafe sexual contact, especially associated with the mucosal trauma of receptive rectal intercourse, transfusion of contaminated blood or blood products and sharing of contaminated needles or transplacental or postpartum transmission from an infected mother to the fetus (by cervical or blood contact at delivery and in breast milk). HIV is not transmitted by casual household or social contact [2].

Human CD4⁺T, lymphocytes, macrophages, microbial, dendritic and langerhans cells are believed to be targets for HIV-1 infection. The main target is CD4⁺T helper cell, a type of T cell; T-cells are an important part of the immune system because they help to facilitate the body's response to many common but potentially fatal infections. By ways that are not yet completely understood, HIV's life cycle directly or indirectly causes a reduction in the number of T-cells in the body, eventually resulting in an increased risk of infections. Over time there are not enough T-cells in the body. At this stage, a person is said to have Acquired Immuno Deficiency Syndrome or AIDS, and becomes susceptible to infectious that a healthy immune system could deal with. The time in between the first infection and initiation of antibody synthesis is 6-12 weeks. The medium time to receive an AIDS diagnosis among those infected with HIV is 7-10 years [1].

We have strong evidences [3] that infectivity of HIV infected is not constant, rather it depends on the stage of infection and viral load. On the basis of most of the studies on this topic the transmission probability is highest at the early stage of infection .Considering the onset time of AIDS from HIV, Mukandavire et.al [4] took into account the time delay in incubation period and expressed incubation period as a function of time t.

Kaplan [5], Greenhalgh [6], Lewis and Greenhalgh [7] and Bobashev et al. [8] developed mathematical models of the spread of HIV by sharing of contaminated needles.

In the present work, we have worked in a larger perspective by constructing a mathematical model of the spread of HIV taking into account unsafe sex, transfusion of contaminated blood or blood products or sharing of contaminated needles and birth and death in the group not infected by HIV as well as in the group infected by HIV which is an advancement of the model proposed by Tapadar and Ghosh [9].

In addition to this, here we concentrate on creating a practical profile by using the prior data to estimate the posterior results in the context of India. Rao [10] identified that there is a lack of scientific means of gathering information about the infectious period of HIV. Godbole and Mehendale [11] presented a vivid profile of spread of

HIV/AIDS in India and about the preventive measures taken. Nagelkerke et al. [12] described a dynamic compartmental simulation model for HIV/AIDS epidemics in Botswana and India developed to identify the best strategies for preventing spread of HIV/AIDS. Chaturvedi [13] proposed a Neuro-Fuzzy approach to develop dynamic model of HIV population in Agra region in India. Though major risk groups have been identified but still the actual picture of the dynamics of HIV in India is not transparent. We have, in our hand, the details of the spread of HIV/AIDS in India for last few years [14, 15, 16 17, 18, 19, 20, 21]. This data is used carefully to forecast the total scenario of the present interest for the coming few years.

Formulation of the Model

To construct the model we assume the following:

I = Number of persons carrying HIV at time t

S = Number of persons not carrying HIV at time t

 P_1 = Mass of the population, initially not infected by HIV, going for unsafe sex

 P_2 = Mass of the HIV infected population going for unsafe sex

 P_3 = Mass, initially not infected but getting infected by HIV due to transfusion of contaminated blood or blood products or by sharing of contaminated needles

b = Birth rate per individual in the group not infected by HIV

d = Death rate per individual in the group not infected by HIV

b' = Birth rate per individual in the group of HIV infected

d' = Death rate per individual in the group of HIV infected.

We consider the governing equations [9]

$$\frac{dS}{dt} = -\alpha P_1 P_2 - \beta P_3 + bS - dS$$

$$\frac{dI}{dt} = \alpha P_1 P_2 + \beta P_3 + b'I - d'I$$
(1)

where α , β are the parameters characterizing the spread of HIV. Again we have,

$$\begin{array}{c}
P_1 = Q_1 S \\
P_2 = Q_2 I \\
P_3 = Q_3 S
\end{array}$$

$$(2)$$

where Q_1 , Q_2 and Q_3 are corresponding proportionality factors such that $Q_i \in (0, 1)$ for i=1,2,3.

Initially let there be 'n' individuals not infected by HIV and 'a' individuals infected by HIV, i.e., S(0) = n, I(0) = a.

So we get,

 $\frac{dS}{dt} = -\alpha Q_1 S Q_2 I - \beta Q_3 S + (b-d)S$

and

$$\frac{dI}{dt} = \alpha . Q_1 S . Q_2 I + \beta . Q_3 S + (b' - d') I$$
(3)

with

S(0) = n, I(0) = a

or,

and

$$\frac{dS}{dt} = -pSI - qS + rS$$

$$\frac{dI}{dt} = pSI + qS + r'I$$
(4)

with

i.e.

$$S(0) = n, I(0) = a$$

where $p = \alpha Q_1 Q_2$, $q = \beta Q_3$, r = b - d and r' = b' - d'.

Search for Equilibrium Points

For equilibrium point, we must have

$$\frac{dS}{dt} = 0 \quad \text{and} \quad \frac{dI}{dt} = 0$$

$$, \quad -pSI - qS + rS = 0$$

$$psI + qS + r'I = 0$$

$$(5)$$

Solving this system, we get the points of equilibrium as (0, 0) and $\left[\frac{r'(q-r)}{pr}, \frac{r-q}{p}\right]$. The second equilibrium point exists only if r > q and r' < 0 to

maintain the non-negative identity of both S and I.

Analysis of Stability of Equilibrium Points

An equilibrium point is considered to be stable if the system always returns to it after small disturbances. If the system moves away from the equilibrium after small disturbances, then the equilibrium is unstable. The number of eigen values is equal to the number of state variables. In our case, there will be two eigen values. If both the eigen values are real, then the equilibrium point is said to be a node and if they are conjugate complex numbers, then it is said to be a focus. If both the eigen values are

64

positive, then the equilibrium point is an unstable node; if both the eigen values are negative, then it is a stable node and if one is positive and the other is negative, then it is a saddle point. For complex eigen values if the real part is positive, then the equilibrium point is an unstable focus and if the real part is negative, then it is a stable focus.

For the present system, we have the characteristic equation as $\begin{vmatrix} -pI - q + r - \lambda & -pS \end{vmatrix} = 0$

$$\begin{vmatrix} -pI - q + r - \lambda & -pS \\ pI + q & pS + r' - \lambda \end{vmatrix} = 0$$

For the equilibrium point (0, 0), we have

$ -q+r-\lambda $	$0 \mid -0$
q	$r'-\lambda\Big ^{-0}$

which gives, $\lambda = r'$, r-q. If r > q, r' > 0 then both the eigen values are positive. In this case, the equilibrium point (0, 0) is an unstable node. If r > q, r' < 0, then it is a saddle point. If r < q, r' > 0 then it is again a saddle point. If r < q, r' < 0, then it is a stable node.

In the domain r > q, r' < 0 for the equilibrium point $\left[\frac{r'(q-r)}{pr}, \frac{r-q}{p}\right]$ we have

the characteristic equation as

,

$$\begin{vmatrix} -\lambda & \frac{-r'(q-r)}{r} \\ r & \frac{r'q}{r} - \lambda \end{vmatrix} = 0$$

which gives

$$\lambda = \frac{\frac{r'q}{r} \pm \sqrt{\frac{{r'}^2 q^2}{r^2} + 4r'(r-q)}}{2}$$

If the discriminant $\frac{r'^2 q^2}{r^2} + 4r'(r-q) \ge 0$ both the eigen values are negative and in

that case, it represents a stable node. Again if the discriminant $\frac{r'^2q^2}{r^2} + 4r'(r-q) < 0$ then the equilibrium point represents a stable focus.

We have framed the series solution of (4) by the following way:

$$S = n + (-pna - qn + rn) t + \frac{1}{2} [(q - r)^{2} n + 2pqan - pr'an - 2aprn + a^{2}p^{2}n - ap^{2}n^{2} - pqn^{2}] t^{2} + \dots$$
and
$$I = a + (pna + qn + r'a) t + \frac{1}{2} [aprn + qrn + 2pr'an + qr'n + r'^{2}a - q^{2}n - 2pqn - a^{2}p^{2}n + ap^{2}n^{2} + pqn^{2}] t^{2} + \dots$$
(6)

Mutual Co-existence of the HIV not-infected and HIV infected individuals: From (4), we get

$$\frac{dI}{dS} = \frac{pSI + qS + r'I}{-pSI - qS + rS} \tag{7}$$

Here we restrict ourselves in the domain r > q, r' < 0. In most of the cases, it is expected that the birth rate per individual is less than the death rate per individual in the group of HIV infected people. So it looks good to assume r' < 0. Also the net rate of increase (rate of birth - rate of death) per individual in the group not infected by HIV is expected to be greater than the rate per individual of getting infected by HIV due to transfusion of contaminated blood or blood products or by sharing of contaminated needles in the same group. So, r > q looks consistent in most of the cases.

Case A:
$$\frac{dI}{dS} > 0$$

Subcase (i): $\frac{dI}{dt} > 0$ and $\frac{dS}{dt} > 0$ i.e., pSI + qS + r'I > 0 and -pSI - qS + rS > 0. In this case, we have $I < \frac{r-q}{p}$. We consider $I = \frac{r-q}{p} - h$ for h > 0. From the inequality pSI + qS + r'I > 0, we have

$$S > \frac{1}{(r-ph)} \left[r'h - \frac{r'(r-q)}{p} \right]$$

provided $h < \frac{r}{p}$. But as I > 0, $f < \frac{r-q}{p} < \frac{r}{p}$, So the above range for S is a consistent range for S in this case.

We take
$$f(h) = \frac{1}{r - ph} \left[r'h - \frac{r'(r - q)}{p} \right]$$

A Dynamical Model of the Spread of HIV/AIDS

Therefore,
$$f'(h) = \frac{r'q}{(r-ph)^2} < 0$$
 since $r' < 0$

So, f(h) is a decreasing function of h. Hence f(h) < f(0) for h > 0, i.e.,

$$\frac{1}{r-ph}\left[r'h-\frac{r'(r-q)}{p}\right] < \frac{r'(q-r)}{pr}$$

Hence in this case, we fail to get any specific range for S.

Subcase (ii): $\frac{dI}{dt} < 0$ and $\frac{dS}{dt} < 0$, i.e., pSI + qS + r'I < 0 and -pSI - qS + rS < 0. In this case, we have $I > \frac{r-q}{p}$. We consider $I = \frac{r-q}{p} + h$ for h > 0. From the inequality pSI + qS + r'I < 0, we have

$$S < \frac{1}{(r+ph)} \left[\frac{r'(q-r)}{p} - r'h \right]$$

We take
$$g(h) = \frac{1}{(r+ph)} \left[\frac{r'(q-r)}{p} - r'h \right]$$

Therefore,
$$g'(h) = -\frac{r'q}{(r+ph)^2} > 0$$
 since $r' < 0$.

So, g (h) is an increasing function of h. Hence g(h) > g(0) for h > 0, i.e., $\frac{1}{\left\lfloor \frac{r'(q-r)}{r'(q-r)} - r'h \right\rfloor} > \frac{r'(q-r)}{r'(q-r)}$

$$\frac{1}{(r+ph)} \left\lfloor \frac{r'(q-r)}{p} - r'h \right\rfloor > \frac{r'(q-r)}{pr}$$

Hence in this case also we fail to get any specific range for S.

Case B:
$$\frac{dI}{dS} < 0$$

Subcase (i): $\frac{dI}{dt} < 0$ and $\frac{dS}{dt} > 0$, i.e., pSI + qS + r'I < 0 and -pSI - qS + rS > 0. In this case, we have $I < \frac{r-q}{p}$ and we take $I = \frac{r-q}{p} - h$ where h > 0. From the inequality pSI + qS + r'I < 0, we get

$$S < \frac{1}{(r-ph)} \left[r'h - \frac{r'(r-q)}{p} \right]$$

provided $h < \frac{r}{p}$. But as I > 0, $h < \frac{r-q}{p} < \frac{r}{p}$. So the above range for S is a consistent range for S in this case. By previous argument in Case A : subcase (i) ,we have

$$\frac{1}{(r-ph)} \left[r'h - \frac{r'(r-q)}{p} \right] < \frac{r'(q-r)}{pr}$$

Hence,
$$S < \frac{r'(q-r)}{pr}$$

Subcase (ii): $\frac{dI}{dt} > 0$ and $\frac{dS}{dt} < 0$, i.e., pSI + qS + r'I > 0 and -pSI - qS + rS < 0. In this case, $I > \frac{r-q}{p}$ and we take $I = \frac{r-q}{p} + h$ where h > 0. From the inequality pSI + qS + r'I > 0, we get

$$S > \frac{1}{(r+ph)} \left\lfloor \frac{r'(q-r)}{p} - r'h \right\rfloor.$$

Now by previous argument in case A : subcase (ii), we have

$$\frac{1}{(r+ph)} \left[\frac{r'(q-r)}{p} - r'h \right] > \frac{r'(q-r)}{pr}$$

Hence, $S > \frac{r'(q-r)}{pr}$.

Sign of <i>dI/dS</i>	Serial No.	Sign of dS/dt	Sign of <i>dI/dt</i>	Range of S	Range of <i>I</i>
+	1	+	+	No specific range for <i>S</i>	$I < \frac{r-q}{p}$
	2	_	_	No specific range for <i>S</i>	$I > \frac{r-q}{p}$
	3	+	_	$S < \frac{r'(q-r)}{pr}$	$I < \frac{r-q}{p}$
	4	-	+	$S > \frac{r'(q-r)}{pr}$	$I > \frac{r-q}{p}$

Keeping in view all the above facts we can have the following table :

Situation 3 is analogous to the model where the group not infected by HIV increases its number, and the group infected by HIV decreases its number. On the other hand, situation 4 is very alarming where the group not infected by HIV reduces its number as well as the group infected by HIV expands with time. This situation can be termed as critically epidemic situation.

Results

Estimation of parameters in the context of India

We work to find out the best fitted values of the parameters p, q, r and r' present in the model in the context of India. For this, we have taken into account the existing statistical database at Indian context [14, 15, 16, 17, 18, 19, 20, 21]. Previously it was thought that around 5.7 million people were living with HIV in India in the year 2007 - more than in any other country. Better data including the results of a national household survey led to a major revision of the prevalence estimate in July 2007 [17, 18]. It was estimated then that around 2.3 million people in India were living with HIV. This entire process leaves some controversies. The problem arises when sentinel surveillance data is used to estimate a country's HIV burden. Sentinel surveillance is not designed to make estimates, but it has been used to provide rough estimates of the HIV burden for many years, for want of a better approach. There are various biases in the existing sampling process. However these biases are not publicly acknowledged, as a result of which the public is misled. In reality, all sentinel sites are in government hospitals, whereas the majority of the people go to private hospitals. Again the samples are from pregnant women and groups with high risk. So no information is available for other men and women outside these groups. Finally if the condition is unevenly distributed in the population, any sample collected from it cannot represent the entire population. So the latest calculation may give us a better profile of the problem, but they too have their limitations [19].

So in the present work for calculations, we have taken into account both these databases and furnished two different sets of values of the parameters involved in our model as well as two different sets of graphical profiles.

Using first set of data:

We have the following table in this connection [14, 15, 16]:

Year	Indian Population	HIV/AIDS population In India
	(in millions)	(In millions)
2005	1095	2.5
2006	1121.8	3.97
2007	1143	5.7

The calculation gives rise to the following best-fitted magnitudes of the parameters:

 $p = 8.75 \times 10^{-10}$, $q = 4.4731 \times 10^{-4}$, r = 0.0446 and r' = --0.6688.

These estimated values can be further used effectively to predict the number of HIV/AIDS infected in the context of India at near future instants.

Using second set of data which Government of India accepts now-a-days: We have the following table in this regard [19, 20, 21]

Year	Indian Population	HIV/AIDS population In India
	(in millions)	(In millions)
2006	1121.8	2
2007	1143	2.3
2008	1148	2.5

The calculation gives rise to the following best-fitted magnitudes of the parameters:

 $p = 2.3367 \times 10^{-10}$, $q = 1.037 \times 10^{-3}$, r = 0.01736 and r' = -0.6842.

We have used these estimated values to predict probable HIV/AIDS population in India in coming future.

Our Forecast for HIV/AIDS Population in India

Using the first set of estimated values of parameters we have estimated the number of HIV infected and uninfected individuals in India upto 2015 (2007 onwards) and in this context we present Figure 1 and Figure 2.



Figure 1 (Total number of uninfected versus time using first set of data)



Figure 2 (Total number of infected versus time using first set of data)

Using the second set of estimated values of the parameters we find the probable values of S and I up to 2015 (2008 onwards) and plotting those values graphically here we present Figure 3 and Figure 4.



Figure 3 (Total number of uninfected versus time using second set of data)



Figure 4 (Total number of infected versus time using second set of data)

It is notable that although the years 2009 and 2010 were already past but still we keep them in the estimation period. This is due to the fact that the corresponding factsheets for these two years are still unavailable. The parameters involved here are very delicate in nature. If we go for a long run, the values of r and r' must be changed significantly. So here we have considered a comparatively small time range (up to the year 2015) where it is very much expected that the magnitudes for these should have a very small amount of variation so that the result is not affected altogether.

Discussion

In the present work, we have proposed a mathematical model of the spread of HIV and estimated the parameters involved in that model in the context of India in order to

make the forecasting of number of infected people in India possible. These parameters can also be estimated for any other domain in the world using the corresponding data for that particular domain. Using the estimated parameters, one may predict what will be the total number of infected individuals by HIV/AIDS at near future for any geographical domain. The present statistical analysis has limitation that it cannot be used for a very large scale of future time because, in the process, values of r and r' will certainly be changed and that will disturb the entire process. To overcome this problem, we have to introduce additional equations in the model which sufficiently depicts the time-variation of these two parameters. Concentrating on this fact, further research can be carried to have the forecasting for a sufficiently large scale of time.

References

- [1] S. Bhattacharjee: Why does HIV manifest into AIDS in humans, *Everyman's Science*, **XLI (6)**, 401 (2007).
- [2] Causes of HIV/AIDS, http://www.wrongdiagnosis.com/h/hiv_aids/causes.htm.
- [3] S. Cassels, S. J. Clark and M. Morris: Mathematical Models for HIV Transmission Dynamics: Tools for Social and Behavioral Science Research, *Acquir. Immune Defic. Syndr*, **47** (1), S34 (2008).
- [4] Z. Mukandavire, W. Garira and C. Chiyaka: Asymptotic properties of an HIV/AIDS model with a time delay: *Journal of Mathematical Analysis and Applications*, **330**, 916 (2006).
- [5] E. H. Kaplan, Needles that kill: Modelling human immunodeficiency virus transmission via shared drug injection equipment in shooting galleries, *Rev.Infect. Dis.*, **11**, 289 (1989).
- [6] D. Greenhalgh: Effects of Heterogeneity on the Spread of HIV/AIDS Among Intravenous Drug Users in Shooting Galleries, *Mathematical Biosciences*, **136**, 141 (1996).
- [7] F. Lewis and D. Greenhalgh: Three stage AIDS incubation period: a worst case scenario using addict-needle interaction assumptions, *Mathematical Biosciences*, **169**, 53 (2001).
- [8] G.V. Bobashev et al.: Geographically-Enhanced Mathematical Models of HIV Dynamics, http://www.xjtek.com/files/papers/geographicallyenhanced2004.pdf,2004 [presented at NIDA Symposium on AIDS, Cancer and Related Problems, May 26, 2004, St. Petersburg, Russia].
- [9] A. Tapadar and K. Ghosh: Spread of HIV: A Mathematical Model, Proceedings of the 13th Asian Technology Conference in Mathematics (ATCM 2008) [Edited by Yang, Majewsk, de Alwis & Khairiree] (Held at the Suan Sunandha Rajabhat University, Bangkok, Thailand during 15-19 December, 2008), 203 (2008).
- [10] A.S.R.S. Rao: Mathematical Modelling of AIDS epidemic in India, *Current Science*, **84** (9), 1192 (2003).

- [11] S. Godbole and S. Mehendale: HIV/AIDS epidemic in India: risk factors, risk behaviour & strategies for prevention & control, *Indian J. Med. Res.*, **121**, 356 (2005).
- [12] N.J.D. Nagelkerke et al.: Modelling HIV/AIDS epidemics in Botswana and India: impact of interventions to prevent transmission, *Bulletin of the World Health Organization*, 80 (2), 89 (2002).
- [13] D.K. Chaturvedi: Dynamic Model of HIV/AIDS Population of Agra Region, Int. J. Environ. Res. Public Health, 2(3), 420 (2005).
- [14] http://data.unaids.org/pub/EPISlides/2007/2007_epiupdate_en.pdf.
- [15] <u>http://content.nejm.org/cgi/content/full/356/11/1089</u>.
- [16] <u>http://www.avert.org/indiaaids.htm</u>.
- [17] UNAIDS: Press Release: 2.5 million people in India living with HIV, according to new estimates, 6th July, 2007.
- [18] UNAIDS: India: Country situation, 2008.
- [19] M.P. Kumar: 20 million or 2 million? http://infochangeindia.org/200801166819/Agenda/HIV/Aids-bigquestion/20million or 2million.html.
- [20] http://www.usaid.gov/locations/asia/countries/india.
- [21] http://www.cia.gov/library/publications/the-world-factbook/print/in.html.