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Two Dimentional Flow in Renal Tubules with Linear Model

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Abstract

Two dimensional flow in renal tubules is studied by considering radial component of velocity as a linear function of z by stream function approach. We have shown that the volume rate of flow is a quadratic function of z. We also calculated the tube radius required for minimum radial velocity. We have also calculated the range for pressure drop for uniform reabsorption rate.

Keywords: Renal flow; Tubules; Nephron; Radial velocity; Reabsorption; Axial velocity; Stream function.

AMS Subject Classification: 76Zxx

Introduction

The functional unit of the kidney is called the Nephron or Renal tubule, each kidney as about 1 million of these tubules. One major part of a nephron is the glomerular tuft trough which blood coming from the renal artery and afferent arterioles is filtered. The glomelecular filtrate is essential identical to plasma and no chemical separation occurs up to this point. If the kidney delivers this filtrate for excretion, the body loses many valuable materials, including water, at a rate faster than the one at which they can be supplied by feeding. Thus 80 percent of filtrate is reabsorbed in the proximal tubule, and of the remaining about 95 percent is further reabsorbed by the end of the collection of the collecting duct. This reabsorb ion creates a radial component of the velocity in the cylindrical tubule, which must be considered along with the axial component of the velocity.

Due to loss of fluid from the walls, both the radial and axial velocities decreases with z. Mathematically, we have to solve problem of flow of a viscous fluid in a circular cylinder when there are axial and radial components of velocity and the radial velocity at all points on the surface of the cylinder is prescribed and is a decreasing function $\phi(z)$ of z.

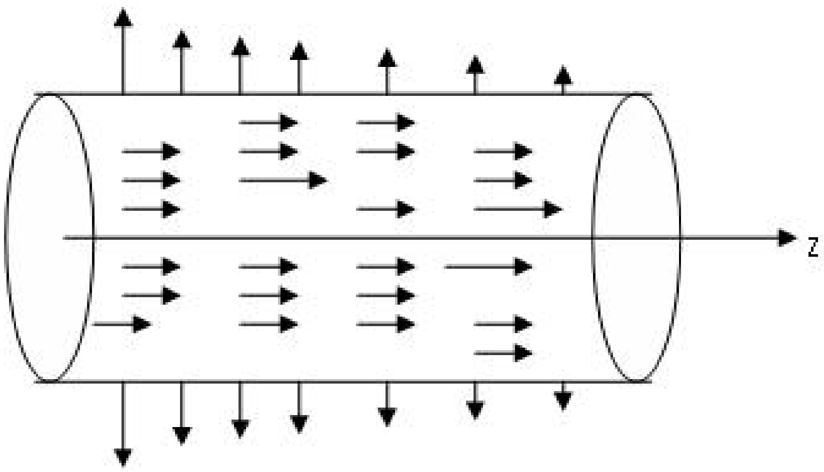

Figure 1: Two-dimensional flow in renal tubule.

During last decades an extensive research work has been done on the fluid dynamics of biological fluids in presence and absence of magnetic fluid due to bioengineering and medical applications [1-3]. Mathematical models of the renal tubule are classified as epithelial, tubular or multitubular. The tubular models allow prediction of the effect of epithelial transport to modify the luminal solution and conversely the effect of altered luminal composition on transepithelial fluxes. Here the conservation equation constitute asset of ODE with initial data that must be integrated along tubule length. The mulitubular models are mathematically the most complex and have character of BVP, where transport along an entire tubule contributes to the interstitial composition. Frank [4] have analyzed the model in which a force of unspecified origin drives the fluid from descending thin limb (DTL) to interstitial vascular space, thus concentrating the solution in DTL Layton [5] have solved hyperbolic partial differential equations of Renal model explicitly for both dynamic and steady state. Layton. et. al. [6] have done Numerical Simulation of propagating concentration profiles in renal tubules. Layton. et. al. [7] gave a dynamic numerical method for models of the urine concentrating mechanism Pitman [8] have studied mass conservation in a dynamic numerical method for a model of the urine concentrating mechanism. Sands. et. al. [9] explained in detail urine concentrating mechanism and its regulation in their book. Marcano. et. al [10] gave an inverse algorithm for a mathematical model of an avian urine concentrating mechanism. Anita et. al. [11] have studied the dynamics of coupled nephrons. Thus many researchers have studied the mathematical model of renal fluid flows.

In this paper we study renal fluid flow by using linear model neglecting the inertial term and considering fluid as nonviscous in cylindrical polar coordinate system.

Basic Equations and Boundary Conditions

At the outset, we may note that the equation of motion can be simplified since the inertial term in relation to the viscous terms can be neglected. The average tubular radius is about $10^{-3}cm,$ the average velocity is about $10^{-1}cm/\sec,$ and fluid velocity about $7\times10^{-3}dynes\sec/cm^{2}$. This gives a Reynolds number of about 10^{-3} and, since this is very much less than 1, we neglect the inertial terms to get the following equations of continuity and motion.

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial v_z}{\partial z}=0, \tag{1}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial r}=\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{\partial^2 v_z}{\partial z^2}, \tag{2}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial z}=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial z})+\frac{\partial^2 v_z}{\partial z^2}. \tag{3}$$

The boundary conditions are

$$\frac{\partial v_z}{\partial z}=0,\ v_r=0,\ v_z=\mathit{finite}, \text{ at r=0.} \tag{4}$$

$$v_z=0,\ v_r=\phi(z) \text{ at r =R,} \tag{5}$$

$$\left.\begin{aligned} p&=p_0 \text{ at z=0}\\ p&=p_L. \end{aligned}\right\} \tag{6}$$

Partially differentiating equations (2) and (3) with respect to z & r respectively and eliminating p we get

$$\frac{\partial^2}{\partial r\partial z}\left[\frac{1}{r}\frac{\partial}{\partial z}(rv_r)\right]+\frac{\partial^3 v_r}{\partial z^3}=2\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]+\frac{\partial^3 v_z}{\partial z^2\partial r}. \tag{7}$$

Partially differentiating equation (7) with respect to z and using equation (1), we get

$$\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\right)\right]+2\frac{\partial}{\partial}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)\right]+\frac{1}{r}\frac{\partial^4}{\partial z^4}\right\}(rv_r)=0. \tag{8}$$

Introducing the stream function we can satisfy the equation (1) defined by

$$v_r=\frac{1}{r}\frac{\partial\psi}{\partial z},\ v_z=-\frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{9}$$

Substituting (9) in (7), we get

$$D^2\left(D^2\psi\right)=0, \tag{10}$$

where D^2 is defined by

$$D^2 = \left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right). \tag{11}$$

If $$v_r = f(r)g(z), \tag{12}$$

then the form of (8) suggests that an analytical solution may be possible.

If $$g(z)= A_0 + A_1 z \tag{13}$$

From equation (5) since $v_r = \phi(z)$ when r=R, we get

$$f(r)g(z)=\phi(z). \tag{14}$$

This suggest that we may get an analytical solution when the radial component of velocity on the surface of the cylinder is given by

$$\phi(z)= a_0 + a_1 z . \tag{15}$$

Radial Velocity at wall Decreases linearly with z

For (10), let the solution be

$$\psi(r,z)=F(r)\left(a_0 z+\tfrac{1}{2}a_1 z^2\right)+G(r) \tag{16}$$

So that using (9), we get

$$v_r = \frac{1}{r}F(r)(a_0 + a_1 z) \tag{17}$$

$$v_z = -\frac{1}{r}F'(r)\left(a_0 z+\tfrac{1}{2}a_1 z^2\right)-\frac{1}{r}G'(r) \tag{18}$$

$$\begin{aligned}D^2\left(D^2\psi\right)&=\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)^2 F(r)(a_0 z+\tfrac{1}{2}a_1 z^2)+\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)^2 G(r)\\&+2a_1\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)F(r)\end{aligned} \tag{19}$$

From equation (10) we get

$$\begin{aligned}&\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)^2 F(r)(a_0 z+\tfrac{1}{2}a_1 z^2)+\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)^2 G(r)\\&+2a_1\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)F(r)=0\end{aligned} \tag{20}$$

From above equation we get

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)^2 F(r)\left(a_0 z+\tfrac{1}{2}a_1 z^2\right)=0 \tag{21}$$

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)^2 G(r)+2a_1\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)F(r)=0 \tag{22}$$

Equation (20) gives

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)H(r)=0 \tag{23}$$

Where

$$H(r)=\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)F(r) \tag{24}$$

Solving (24) we get

$$\frac{1}{r}\frac{d^2F}{dr^2}-r\frac{dF}{dr}=Ar^4+Br^2 \tag{25}$$

Solving above differential equation we get

$$F(r)=C+Dr^2+\frac{Ar^4}{8}+\frac{Br^2}{2}\ln r \tag{26}$$

From equation (22) we get

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)\left[\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)G(r)+2a_1F(r)\right]=0 \tag{27}$$

Solving equation (27) we get

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)G(r)+2a_1F(r)=Mr^2+N \tag{28}$$

Now from boundary conditions (4) and (5) we have

$$\frac{d}{dr}\left[\frac{1}{r}F'(r)\right]=0 \text{ at r =0} \tag{29}$$

$$\frac{d}{dr}\left[\frac{1}{r}G'(r)\right]=0 \text{ at r =0} \tag{30}$$

$$\frac{1}{r}F(r)=0 \text{ at r = 0} \tag{31}$$

$$\frac{1}{r}F'(r)=0 \text{ and } \frac{1}{r}G'(r)=0 \text{ are finite at r = 0} \tag{32}$$

$$F'(r)=0 \quad G'(r)=0 \quad F(R)=R \tag{33}$$

From (26), (31) and (32) we get

$$C=0,\ B=0 \tag{34}$$

From (26), (32) and (34) we get

$$2DR+\tfrac{1}{2}AR^3=0,\quad DR^2+\tfrac{1}{8}AR^4=R, \tag{35}$$

Solving above equation we get

$$A=\frac{-8}{R^3},\quad D=\frac{2}{R}, \tag{36}$$

Equation (26) becomes

$$F(r)=R[2(\frac{r}{R})^2-(\frac{r}{R})^4] \tag{37}$$

Substituting (37) in (28), we get

$$\frac{d^2G}{dr^2}-\frac{1}{r}\frac{dG}{dr}=Mr^2+N-4a_1\frac{r^2}{R}+2a_1\frac{r^4}{R^3} \tag{38}$$

Integrating (38), we obtain

$$G(r)=N_1+M_1r^2+\frac{Mr^4}{8}+\frac{Nr^2}{2}\ln r-\frac{a_1}{2}\frac{r^4}{R}+\frac{a_1}{12}\frac{r^6}{R^3} \tag{39}$$

From (32) and (39)

$$N=0 \tag{40}$$

From (33) and (39)

$$2M_1R+\frac{MR^3}{2}-\frac{3a_1}{2}R^2=0 \tag{41}$$

Equation (41) can determine only one of the two unknown constants M and M_1, to determine both of these, we need one more relation. This relation can be found in terms of Q_0 which is the total flux at z=0.

Using

$$Q(z)=\int_0^R 2\pi r v_z(r,z)dr \tag{42}$$

$$= 2\pi\int_0^R \left[\left(\frac{4r^3}{R^3} - \frac{4r}{R}\right)\left(a_0 z + \tfrac{1}{2}a_1 z^2\right) - 2M_1 r - \frac{Mr^3}{2}\frac{2a_1}{R}r^3 + \frac{a_1 r^5}{2R^3} \right] dr \quad (43)$$

$$\frac{Q_0}{2\pi R^2} = \frac{MR^2}{8} - \frac{a_1}{3}R \quad (44)$$

$$M = \frac{8}{R^2}\left(\frac{Q_0}{2\pi R^2} + \frac{a_1}{3}R\right) \quad (45)$$

$$M_1 = -\frac{Q_0}{\pi R^2} + \frac{a_1}{12}R \quad (46)$$

From (39), (40), (45) and (46) we get

$$G(r) = \left(\frac{a_1 R}{12} - \frac{Q_0}{\pi R^2}\right)r^2 + N_1 + \frac{1}{R^2}\left(\frac{Q_0}{2\pi R^2} + \frac{a_1 R}{12}\right)r^4 - \frac{a_1}{2}\frac{r^4}{R} + \frac{a_1}{12}\frac{r^6}{R^3} \quad (47)$$

The constant N_1 need not be determined since $\psi(r,z)$ can always contain an arbitrary constant without affecting the velocity components.

From (17), (37) and (47) we get

$$v_r(r,z) = \left[2\frac{r}{R} - \left(\frac{r}{R}\right)^3\right](a_0 + a_1 z) \quad (48)$$

$$v_z(r,z) = \left(1 - \frac{r^2}{R^2}\right)\left[\frac{2Q_0}{\pi R^2} - \frac{2}{R}\left(a_0 z + \tfrac{1}{2}a_1 z^2\right) - \frac{a_1 R}{2}\left(\tfrac{1}{3} - \frac{r^2}{R^2}\right)\right] \quad (49)$$

Differentiating equation (43) we get

$$\frac{dQ}{dz} = 8\pi\left(a_0 z + \tfrac{1}{2}a_1 z^2\right)\int_0^R \left(\frac{r^3}{R^3} - \frac{r}{R}\right)dr = -2\pi R(a_0 + a_1 z) \quad (50)$$

Thus the decrease of flux is equal to the amount of the fluid coming out of the cylinder per unit length per unit time.

Integrating equation (50) we get

$$Q(z) = Q_0 - \pi R(2a_0 z + a_1 z^2) \quad (51)$$

From equation (49) and (51) we get

$$v_z = \left(1 - \frac{r^2}{R^2}\right)\left[\frac{2Q(Z)}{\pi R^2} - \frac{a_1 R}{2}\left(\tfrac{1}{2} - \frac{r^2}{R^2}\right)\right] \quad (52)$$

Using (2), (3), (48), (49) and (51), we get

$$\frac{\partial p}{\partial r} = -\frac{8\mu r}{R^3}(a_0 + a_1 z) \quad (53)$$

$$\frac{\partial p}{\partial z} = -\frac{4a_1\mu}{R}[\frac{r^2}{R^2} + \frac{2Q(z)}{a_1\pi R^3} + \tfrac{1}{3}] \tag{54}$$

Integrating equation (53) we get

$$p(r,z) = -\frac{4a_1\mu r^2}{R^3}(a_0 + a_1 z) + k(z) \tag{55}$$

Differentiating (55) partially with respect to z and using (54) we get

$$k'(z) = -\frac{4a_1\mu}{R}[\tfrac{1}{3} + \frac{2Q(z)}{a_1\pi R^3}] \tag{56}$$

Integrating above equation we get

$$k(z) = -\frac{4a_1\mu}{R}[\tfrac{1}{3}z + \frac{2z\overline{Q}(z)}{a_1\pi R^3}] + k_0 \tag{57}$$

where

$$\overline{Q}(z) = \int_0^z Q(z)dz$$

The average pressure p(z) at any section is given by

$$\overline{p}(z) = \frac{\int_0^r p(r,z)2\pi r dr}{\int_0^r 2\pi r dr} \tag{58}$$

$$= -\mu[\frac{2a_0}{R} + (\frac{8\overline{Q}(z)}{\pi R^4} + \frac{10a_1}{3R})z] \tag{59}$$

Thus the pressure drop over tube length L is

$$\Delta\overline{p} = \overline{p}(0) - \overline{p}(L)$$

Using equation (59) we get

$$\overline{p}(z) = \mu(\frac{8\overline{Q}(L)}{\pi R^4} + \frac{10a_1}{3R})L \tag{60}$$

The Maximum and Minimum of Radial Velocity

We have a condition that the Maximum and Minimum of an equation is given by

$$\left.\begin{array}{l}\text{If } f'(c)=0 \text{ and } f''(c)<0\\ \text{Then } f(x) \text{ has a maximum at x=c}\\ \text{If } f'(c)=0 \text{ and } f''(c)>0\\ \text{Then } f(x) \text{ has a minimum at x=c}\end{array}\right\} \quad (61)$$

Differentiating equation (48) with respect to r we get

$$\frac{dv_r}{dr} = [\frac{2}{R} - \frac{3r^2}{R^3}](a_0 + a_1 z) \quad (62)$$

From the condition (61) equation (62) becomes

$$[\frac{2}{R} - \frac{3r^2}{R^3}] = 0 \quad (63)$$

From equation (63) we get

$$\frac{r}{R} = \pm\sqrt{\frac{2}{3}} \quad (64)$$

Differentiate again equation (62) we get

$$\frac{d^2 v_r}{dr^2} = -\frac{6r}{R^3}(a_0 + a_1 z) \quad (65)$$

Clearly $\frac{d^2 v_r}{dr^2} < 0$ when $\frac{r}{R} = \sqrt{\frac{2}{3}}$

Then Maximum Radial velocity exists at $\frac{r}{R} = \sqrt{\frac{2}{3}}$

Then Maximum Radial Velocity is given by

$$(v_r)_{\max} = [2\sqrt{\tfrac{2}{3}} - (\sqrt{\tfrac{2}{3}})^3](a_0 + a_1 z)$$

$$(v_r)_{\max} = \tfrac{4}{3}\sqrt{\tfrac{2}{3}}(a_0 + a_1 z) \quad (66)$$

Then from equation (65) it is clear that

$$\frac{d^2 v_r}{dr^2} > 0 \text{ when } \frac{r}{R} = -\sqrt{\frac{2}{3}}.$$

Then the Minimum radial velocity exists at $\frac{r}{R} = -\sqrt{\frac{2}{3}}$

$$(v_r)_{\min} = [2(-\sqrt{\tfrac{2}{3}}) - (-\sqrt{\tfrac{2}{3}})^3](a_0 + a_1 z)$$

$$(v_r)_{\min} = -\tfrac{4}{3}\sqrt{\tfrac{2}{3}}(a_0 + a_1 z) \tag{67}$$

The Maximum and Minimum of Axial Velocity

Differentiating equation (49) we get

$$\frac{dv_z}{dz} = \tfrac{-2}{R}(1 - \frac{r^2}{R^2})(2a_0 + 2a_1 z) \tag{68}$$

From the condition (61), equation (68) gives

$$2a_0 + 2a_1 z = 0 \tag{69}$$

$$z = -\frac{a_0}{a_1} \tag{70}$$

And differentiating equation (68) with respect to z we get

$$\frac{d^2 v_z}{dz^2} = \tfrac{-4a_1}{R}(1 - \frac{r^2}{R^2})\,. \tag{71}$$

We get the four conditions for the Maximum and Minimum

Case1: If $a_1 > 0,$ and $\left(\frac{r}{R}\right)^2 < 1$

Then $\frac{d^2 z}{dz^2} < 0$, clearly v_z has Maximum at $z = -\frac{a_0}{a_1}$

Case2: If $a_1 > 0,$ and $\left(\frac{r}{R}\right)^2 > 1$

Then $\frac{d^2 z}{dz^2} > 0$, clearly v_z has Minimum at $z = -\frac{a_0}{a_1}$

Case3: If $a_1 < 0,$ and $\left(\frac{r}{R}\right)^2 < 1$

Then $\frac{d^2 z}{dz^2} > 0$, clearly v_z has Minimum at $z = -\frac{a_0}{a_1}$

Case4: If $a_1 > 0,$ and $\left(\frac{r}{R}\right)^2 > 1$

Then $\frac{d^2z}{dz^2} < 0$, clearly v_z has Maximum at $z = -\frac{a_0}{a_1}$

The Pressure drop For Uniform Reabsorption rate over the Tube Length L

The pressure drop over the tube length L is

$$\Delta\overline{p} = \mu\left(\frac{8\overline{Q}(L)}{\pi R^4} + \frac{10a_1}{3R}\right)L \tag{72}$$

For the uniform reabsorption rate, the pressure drop over the tube length L is given by

$$\Delta\overline{p} = \mu\frac{8\overline{Q}(L)}{\pi R^4} \tag{73}$$

Where

$$\overline{Q}(z) = \int_0^z Q(z)dz$$

For the uniform reabsorption rate $Q(z)$ is given by

$$Q(z) = Q_0 - 2a_0\pi Rz \tag{74}$$

Then $\overline{Q}(z)$ is given by

$$\overline{Q}(z) = Q_0 z - a_0\pi Rz^2 \tag{75}$$

From equation (73) we get

$$\overline{Q}(z) = Q_0 z - \frac{[Q_0 - Q(z)]}{2}z$$

$$\overline{Q}(z) = \frac{[Q_0 + Q(z)]}{2}z \tag{76}$$

And

$$Q(0) = Q_0 \text{ At } z = 0$$

From equation (73) we get

$$\Delta\overline{p} = \frac{8\mu}{\pi R^4}\frac{[Q(0) + Q(L)]}{2}L \tag{77}$$

From equation (77) it is clear that

$$\Delta\bar{p} \geq \frac{4\mu Q(0)}{\pi R^4} L \tag{78}$$

From equation (74) at z=L

$$Q(z) = Q_0 - 2a_0 \pi RL \tag{79}$$

From equation (77) and (79)

$$\Delta\bar{p} = \frac{8\mu L}{\pi R^4} Q(0) - 2\pi R a_0 z \tag{80}$$

Clearly from Equation (80) we get

$$\Delta\bar{p} \leq \frac{8\mu L}{\pi R^4} Q(0) \tag{81}$$

From equation (78) and (81) we get

$$\frac{4\mu Q(0)}{\pi R^4} L \leq \Delta\bar{p} \leq \frac{8\mu L}{\pi R^4} Q(0) \tag{82}$$

The pressure drop for the uniform reabsorption if

$$R = 10^{-3} cm,\ Q_0 \approx 4 \times 10^{-7} cm^3 / \sec,$$

$$\mu = 7 \times 10^{-3} dyne \sec/ cm^2,\ L = 1cm,\ Q(L) = 0.2Q(0)$$

is given by

$$\Delta\bar{p} = \frac{8\mu}{\pi R^4} \frac{[Q(0) + Q(L)]}{2} L$$

Substituting the given values we get

$$\Delta\bar{p} = \frac{1.34}{\pi} 10^4 dyne / cm^2$$

Discussion

In this paper we study two dimensional flows in renal tubules with linear model. We consider radial component of velocity as a linear function of z. We calculate the stream function and axial velocity. Using this we calculate volume rate of flow which is a quadratic function of z. We also calculate average flux and pressure drop over the tube length. Using the condition of maximum and minimum of a function, radial velocity will be minimum at $\frac{r}{R} = -\sqrt{\frac{2}{3}}$ and axial velocity depending on the constants a_0 and a_1. For uniform reabsorption rate we calculate the pressure drop and flux of the flow, which is given by the relation (82).

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